PERSISTENCE OF HÖLDER CONTINUITY
FOR NON-LOCAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we consider non-local integro-differential equations under certain natural assumptions on the kernel, and obtain persistence of Hölder continuity for their solutions. In other words, we prove that a solution stays in C^β for all time if its initial data lies in C^β. This result has an application for a fully non-linear problem, which is used in the field of image processing. The proof is in the spirit of [18] where Kiselev and Nazarov established Hölder continuity of the critical surface quasi-geostrophic (SQG) equation.

1. INTRODUCTION AND THE MAIN RESULT

Let $N \geq 1$ be any dimension. We consider the following evolution equation

$$(1) \qquad \partial_t w(t,x) = \int_{\mathbb{R}^N} [w(t,y) - w(t,x)]K(t,x,y)dy$$

where K satisfies the *weak-(∗)-kernel* condition, which will be given in Definition 1.2. The above integral is understood in the sense of principal value. More precisely, we denote the integral operator T_t^K and $(T_t^K)_\epsilon$ for $\epsilon > 0$ corresponding to any given kernel K at time t by

$$(T_t^K)_\epsilon(f)(x) = \int_{|x-y| \geq \epsilon} [f(x) - f(y)]K(t,x,y)dy \quad \text{and}$$

$$(T_t^K)(f)(x) = \lim_{\epsilon \to 0}(T_t^K)_\epsilon(f)(x).$$

Then, (1) is equivalent to $(\partial_t w)(t,x) + T_t^K(w(t,\cdot))(x) = 0$. Related to the above singular integral, there have been many interests recently, not only from the field of analysis, but also from the field of probability (e.g. Caffarelli and Silvestre [6], Schwab [24], Bass and Levin [2], Jacob, Potrykus, and Wu [16], and Chen, Kim, and Kumagai [9]).

Our main concern is to obtain *a priori* estimate for solutions of (1). The aim is to prove the result [4] of Caffarelli, Chan, and Vasseur with different techniques (a similar result for the stationary case was obtained by Kassmann in [17]). In particular, we prove persistence of Hölder continuity in $L^\infty(0, \infty; C^\beta(\mathbb{R}^N))$, which is a new result, by observing the evolution of a dual class of test functions. This class, which appears in the work of Kiselev and Nazarov [18], plays a similar role of the dual space of C^β. They obtained, in [18], Hölder regularity for solutions of the critical

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surface quasi-geostrophic (SQG) equation. It is interesting to compare this method with that of Caffarelli and Vasseur [7]. In [7], the estimate $C^\beta([t,\infty) \times \mathbb{R}^N))$ for any $t > 0$ was proved by using a De Giorgi iteration technique (for other different proofs, we refer to Kiselev, Nazarov, and Volberg [19] and Constantin and Vicol [10]).

We define the $(*)$-kernel condition on the kernel K.

Definition 1.1. Let $0 < \alpha < 2, 0 < \zeta \le \infty, 0 \le \omega < \alpha$ and $1 \le \Lambda < \infty$ (for the case $\alpha \ge 1$, three more parameters ν, s_0 and τ, which are satisfying $(\alpha - 1) < \nu < 1$, $0 < s_0 \le \infty, 0 \le \tau < \infty$ and $\nu + \omega < \min\{N, \alpha\}$, are needed). Then we say that a measurable function $K : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ satisfies the $(*)$-kernel condition on $[0, T]$ for the parameter set $\{\alpha, \zeta, \omega, \Lambda\}$ (if $\alpha \ge 1$, for the parameter set $\{\alpha, \zeta, \omega, \Lambda, \nu, s_0, \tau\}$) if the following conditions hold for all finite $t \in [0, T]$:

(2) $\odot$ Symmetry in x, y: $K(t, x, y) = K(t, y, x)$, for $x, y \in \mathbb{R}^N$.

(3)
$\odot$ Bounds: $\Lambda^{-1} \cdot \mathbf{1}_{|x-y| \le \zeta} \le K(t, x, y)|x-y|^{N+\alpha} \le \Lambda \cdot (1+|x-y|^\omega)$ for $x, y \in \mathbb{R}^N$.

For convenience, we define the associated function k by $k(t, x, z) = K(t, x, x + z)|z|^{N+\alpha}$. Then the above two conditions are equivalent to $k(t, x, y-x) = k(t, y, x-y)$ and $\Lambda^{-1} \cdot \mathbf{1}_{|z| \le \zeta} \le k(t, x, z) \le \Lambda \cdot (1 + |z|^\omega)$, respectively.

Only when $\alpha \ge 1$, we assume one more condition:
(4)
$\odot$ Local Hölder continuity: $\displaystyle\sup_{|z-\tilde{z}| \le s_0, |z| \le s_0} \frac{|k(t, x, z) - k(t, x, \tilde{z})|}{|z - \tilde{z}|^\nu} \le \tau,$ for $x \in \mathbb{R}^N$.

We present the definition of the *weak*-$(*)$-kernel condition, which is slightly weaker than the above $(*)$-kernel condition in Definition 1.1.

Definition 1.2. Under the same setting of the parameters in Definition 1.1, we say that K satisfies the *weak*-$(*)$-kernel condition on $[0, T]$ if K satisfies (2) and (3) for the case $\alpha < 1$. If $\alpha \ge 1$, we ask K to hold the following condition (5) as well as (2) and (3).

(5) $\odot$ Cancellation: $\displaystyle\left| \int_{S^{N-1}} k(t, x, s\sigma)\sigma d\sigma \right| \le \tau \cdot s^\nu$ for $s \in (0, s_0)$ and for $x \in \mathbb{R}^N$

where σ is a surface element on the unit sphere $S^{N-1} \subset \mathbb{R}^N$.

Remark 1.1. The $(*)$-kernel condition in Definition 1.1 implies the *weak*-$(*)$-kernel condition in Definition 1.2. Indeed, for the case $\alpha < 1$, they are exactly same. If $\alpha \ge 1$, then the only difference between them is that the $(*)$-kernel condition needs (4) while the *weak*-$(*)$-kernel condition requires (5). Also, it is easy to verify that (4) implies (5) up to a constant by the following argument: for any $s \in (0, s_0/2)$,

$$\left| \int_{S^{N-1}} k(t, x, s\sigma)\sigma d\sigma \right| = \left| \int_{S^{N-1}_+} k(t, x, s\sigma)\sigma d\sigma + \int_{S^{N-1}_-} k(t, x, s\sigma)\sigma d\sigma \right|$$

$$\le \int_{S^{N-1}_+} \left| k(t, x, s\sigma) - k(t, x, -s\sigma) \right| d\sigma.$$

where S_+^{N-1} and S_-^{N-1} are the upper and the lower hemispheres, respectively. Then, thanks to (4), we have

$$\leq \int_{S_+^{N-1}} \tau \Big|2s\sigma\Big|^\nu d\sigma \leq (C\tau) \cdot s^\nu.$$

Remark 1.2. In the work of [4], the upper bound for k is just Λ while, in this paper, we have $\Lambda \cdot (1 + |x - y|^\omega)$ in (3), which is slightly more general than that of [4]. Some examples with the upper bound (3) can be found in Section 4 of Komatsu [21].

Remark 1.3. The purpose of the condition (5) with $(\alpha - 1) < \nu$ is to consider $T_t^K(f)(\cdot)$ not only as a distribution but also as a locally integrable function. In general, without such an additional cancellation condition, if $\alpha \geq 1$, then $T_t^K(f)(x)$ is not well-defined even for $f \in C_c^\infty$. In Lemma 2.2 and Lemma 2.1, it will be shown that as long as the corresponding kernel K satisfies the *weak*-(∗)-kernel condition, the operator T_t^K is well defined, and $T_t^K(f)$ is a locally integrable function for some class of functions f.

Remark 1.4. Let the kernel K satisfy the *weak*-(∗)-kernel condition for some $\alpha \geq 1$. Then we can combine the two conditions (3) and (5) in order to get an estimate of the integral in (5) for all $s \in (0, \infty)$. Indeed, the condition (3) implies that, for $s \in [s_0, \infty)$,

$$\int_{S^{N-1}} \Big|k(t, x, s\sigma)\Big| d\sigma \leq C\Lambda \cdot (1 + s^\omega) \leq (C\Lambda \cdot s_0^{-\nu}) \cdot s^\nu \cdot (1 + s^\omega).$$

Thus, together with the condition (5), we have, for $s \in (0, \infty)$,

$$(6) \qquad \Big|\int_{S^{N-1}} k(t, x, s\sigma)\sigma d\sigma\Big| \leq \bar{\tau} \cdot s^\nu \cdot (1 + s^\omega)$$

where $\bar{\tau} := \max\{\tau, (C\Lambda \cdot s_0^{-\nu})\}$.

Remark 1.5. We present some typical examples satisfying either the (∗)-kernel condition or the *weak*-(∗)-kernel condition.

(I) For the simplest example, if $K := c_\alpha / |x - y|^{N+\alpha}$ (i.e. $k := c_\alpha$), then the equation (1) becomes the fractional heat equation (some regularity results can be found in Caffarelli and Figalli [5]). This kernel satisfies the (∗)-kernel condition. Indeed, (2) and (3) are trivial. For $\alpha \geq 1$, since k is a constant function, (4) holds for any $\nu \in (\alpha - 1, 1)$ with $\tau = 0$ and $s_0 = \infty$.

(II) One may assume that the kernel has the form not of $K(t, x, y)$ but of $K(t, x - y)$ (for more general cases, we refer to Silvestre [25]). Then the natural symmetry we would impose to the kernel is $K(t, x - y) = K(t, y - x)$, which implies (2) directly. This K holds the *weak*-(∗)-kernel condition for any $\alpha \in (0, 2)$ once we assume the bounds condition (3). Indeed, for $\alpha \geq 1$, the integral in (5) is always zero due to the cancellation from the symmetry (i.e. any $\nu \in (\alpha - 1, 1)$ with $\tau = 0$ and $s_0 = \infty$ works).

Here is our main theorem about persistence of Hölder continuity.

Theorem 1.1. *Let* $\{\alpha, \zeta, \omega, \Lambda\}$ *(for the case* $\alpha \geq 1$, $\{\alpha, \zeta, \omega, \Lambda, \nu, s_0, \tau\}$*) be a set of the parameters in Definition 1.1. Then there exist two constants* $\beta > 0$ *and* $C > 0$ *with the following two properties* (I) *and* (II):

(I). *Let* $w_0 \in (L^1 \cap L^\infty)(\mathbb{R}^N)$ *be a given function and let* $0 < T \leq \infty$. *Let* K *satisfy the* $(*)$*-kernel condition on* $[0, T]$ *(see Definition 1.1). Then there exists a weak solution* w *of* (1) *on* $(0, T)$ *satisfying the following estimates for a.e.* $t \in (0, T)$:

$$(7) \qquad \|w(t, \cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C \cdot \|w_0\|_{C^\beta(\mathbb{R}^N)} \quad if\ w_0 \in C^\beta(\mathbb{R}^N),$$

$$(8) \qquad \|w(t, \cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C \cdot \max\{1, \frac{1}{t^{\beta/\alpha}}\} \cdot \|w_0\|_{L^\infty(\mathbb{R}^N)}, \quad and$$

$$(9) \qquad \|w(t, \cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C\Big(\|w_0\|_{L^\infty(\mathbb{R}^N)} + \max\{1, \frac{1}{t^{(N+\beta)/\alpha}}\} \cdot \|w_0\|_{L^1(\mathbb{R}^N)}\Big).$$

(II). *Let* $w_0 \in C^\infty(\overline{\mathbb{R}^N})$ *be a given function and let* $0 < T \leq \infty$. *Let* K *satisfy the weak-*$(*)$*-kernel condition on* $[0, T]$ *(see Definition 1.2). In addition, we assume*

$$(10) \qquad k(\cdot, \cdot, \cdot) \in C^\infty_{t,x,z}\Big([0, T] \times \overline{\mathbb{R}^N} \times \overline{\mathbb{R}^N}\Big).$$

Suppose that $w \in L^\infty(0, T; L^2(\mathbb{R}^N))$ *is a smooth solution of* (1) *on* $[0, T) \times \mathbb{R}^N$ *for the initial data* w_0. *Then,* w *satisfy* (7), (8), *and* (9) *for any* $t \in (0, T)$.

We concentrate our effort first to prove the part (II) of the above theorem in Section 2, 3, and 4. In fact, we will show the part (II) carefully to ensure that the two constants C and β in the conclusion of the part (II) depend only on the parameters in Definition 1.1. Thus, these two constants C and β depend neither on T nor on any actual norms coming from the smoothness assumption (10). As a result, the part (I), which will be proved in Appendix, follows the part (II) by a limit argument. Unfortunately, if $\alpha \geq 1$, then we need the condition (4), which is more restrictive than (5).

Remark 1.6. More precisely, the conclusion of the part (I) follows once we regularize the function k in a proper way, which should keep all the parameters. In short, since k may not be bounded due to (3), we make it bounded first. Then take a convolution with a mollifier. This process does not hurt the parameter set essentially if $\alpha < 1$. However for the case $\alpha \geq 1$, the cancellation condition (5) is not preserved during the process. That is the reason we impose the $(*)$-kernel condition to the part (I) of Theorem 1.1 instead of the *weak*-$(*)$-kernel condition.

Remark 1.7. Thanks to the symmetry condition (2), we use the following weak formulation of (1):

$$\int_{\mathbb{R}^N} (\partial_t w)(t, x)\eta(x)dx + \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} [w(t, x) - w(t, y)][\eta(x) - \eta(y)]K(t, x, y)dydx = 0$$

for $\eta(\cdot) \in C^\infty_c(\mathbb{R}^N)$ and for a.e. t (e.g. see [4] or [17]).

As in [4], we show how our result can be applied to a fully non-linear problem. We introduce the following non-linear evolution problem:

$$(11) \qquad \partial_t \theta(t,x) - \int_{\mathbb{R}^N} \phi'(\theta(t,y) - \theta(t,x))G(y-x)dy = 0.$$

This equation can be considered as the evolution problem coming from the Euler-Lagrange equation for the variational integral

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \phi(\theta(t,y) - \theta(t,x))G(y-x)dydx$$

(for more detailed explanation, see [4]). This non-linear problem can be found in Giacomin, Lebowitz, and Presutti [14], or in the field of image processing (e.g. see Gilboa and Osher [15], Lou, Zhang, Osher, and Bertozzi [23]).

We impose the following conditions to the equation (11).

Let $0 < \alpha < 2$, $0 < \zeta \le \infty$, $0 \le \omega < \alpha$ and $1 \le \Lambda < \infty$ (for the case $\alpha \ge 1$, we need two more parameters ν and M such that $(\alpha - 1) < \nu < 1$, $0 \le M < \infty$, and $\nu + \omega < \min\{N, \alpha\}$). Let $\phi : \mathbb{R} \to [0, \infty)$ be an even function of class C^2 satisfying

$$\phi(0) = 0 \quad \text{and} \quad \sqrt{\Lambda^{-1}} \le \phi''(x) \le \sqrt{\Lambda}, \quad x \in \mathbb{R}.$$

We assume that the kernel $G : \mathbb{R}^N/\{0\} \to [0, \infty)$ satisfies $G(-x) = G(x)$ and

$$(12) \qquad \sqrt{\Lambda^{-1}} \cdot \mathbf{1}_{|x| \le \zeta} \le G(x) \cdot |x|^{N+\alpha} \le \sqrt{\Lambda} \cdot (1 + |x|^\omega) \text{ for } x \in \mathbb{R}^N/\{0\}.$$

Let $\theta_0 \in (W^{1,1} \cap W^{1,\infty})(\mathbb{R}^N)$ be a given function. For the case $\alpha \ge 1$, we assume further

$$\phi'' \in C^\nu(\mathbb{R}) \text{ with } [\phi'']_{C^\nu(\mathbb{R})} \cdot \|\nabla\theta_0\|^\nu_{L^\infty(\mathbb{R}^N)} \le M.$$

Remark 1.8. The upper bound (12) for $G(\cdot)$ is $\sqrt{\Lambda} \cdot (1 + |x|^\omega)$ and it is more flexible than that of [4], where just $\sqrt{\Lambda}$ was used as the upper bound.

Following the approach of [4], we present the following important consequence of the part (II) of Theorem 1.1.

Theorem 1.2. *We have two constants $\beta > 0$ and $C > 0$ which depend only on the above parameters, and there exists a global-time weak solution θ of the equation (11) with the following estimates for a.e. $t \in (0, \infty)$:*

$$(13) \qquad \|\nabla\theta(t, \cdot)\|_{C^\beta(\mathbb{R}^N)} \le C \cdot \|\nabla\theta_0\|_{C^\beta(\mathbb{R}^N)} \quad \text{if} \quad \nabla\theta_0 \in C^\beta(\mathbb{R}^N),$$

$$\|\nabla\theta(t, \cdot)\|_{C^\beta(\mathbb{R}^N)} \le C \cdot \max\{1, \frac{1}{t^{\beta/\alpha}}\} \cdot \|\nabla\theta_0\|_{L^\infty(\mathbb{R}^N)}, \quad \text{and}$$

$$\|\nabla\theta(t, \cdot)\|_{C^\beta(\mathbb{R}^N)} \le C\Big(\|\nabla\theta_0\|_{L^\infty(\mathbb{R}^N)} + \max\{1, \frac{1}{t^{(N+\beta)/\alpha}}\} \cdot \|\nabla\theta_0\|_{L^1(\mathbb{R}^N)}\Big).$$

The main idea of the above theorem 1.2 is the following: First, we regularize θ_0, G, and ϕ in a proper way so that we obtain a sequence of smooth solutions of (11). Then, we take a derivative ($w := D_e\theta$) to the non-linear equation (11), and we freeze some coefficients. As a result of this process, we obtain the linear equation (1) together with the *weak*-($*$)-kernel condition on the K satisfying (10). Thus, we can use the conclusion of the part (II) of Theorem 1.1. Finally, we extract a weak

solution by a limit argument. This proof will be given in Appendix.

Now we want to explain the main idea of the part (II) of Theorem 1.1, which is the heart of this paper. As mentioned earlier, our proof follows the spirit of the paper [18]. First, thanks to the duality of the equation (1) from the symmetry condition (2), we can focus only on the evolution of $\mathcal{U}_r$, a class of test functions, which is related to the dual space of the Hölder space C^β (see Definition 2.3 of $\mathcal{U}_r$, Lemma 2.4, and Lemma 2.5). In this paper, we take the same definition of the class $\mathcal{U}_r$ from the paper [18] while other classes can be found in Dabkowski [12] and Chamorro [8]. In particular, the class introduced in [12] is quite different from that of [18] and it was successfully used to obtain eventual regularity of the super-critical surface quasi-geostrophic (SQG) equation.

Second, we prove the short-time evolution of test functions (Proposition 3.1). In order to obtain it, we need to manage the competition (refer to Remark 3.4) between the L^p condition and the concentration condition. The former condition, which can be proved from the lower bound $\Lambda^{-1} \cdot \mathbf{1}_{|z| \leq \zeta}$, of the kernel has a regularization effect (Lemma 3.4, Lemma 3.5) as a diffusion term in usual PDEs does. However, the latter condition comes from the upper bound $\Lambda \cdot (1 + |z|^\omega)$ of the kernel and this upper bound plays a similar role as a source term in usual PDEs (Lemma 3.3).

In addition, since the length of the time interval coming from the conclusion of Proposition 3.1 is proportional to r^α where r is the parameter of $\mathcal{U}_r$, it should be verified that we can repeat the short-time evolution (Proposition 3.1) as many times as we want in order to reach any fixed time (refer to Remark 4.1).

For the case $\alpha < 1$, the main difficulty is to handle both lower and upper bounds (3) of the kernel: in particular, both the finite size ζ of support of the lower bound and the term $(1 + |z|^\omega)$ of the upper bound cause some troubles. In order to cover the case $\alpha \geq 1$, we use the cancellation condition (5), which is designed to cancel desirable amount of singularity at $x = y$ of the kernel. Then we can interpret $T_t^K(f)$ as locally integrable functions for some class of functions (see Lemma 2.1, Lemma 2.2). This fact will be crucial to prove the concentration condition (Lemma 3.3).

We want to mention a few articles related to the integral operator T_t^K corresponding a kernel K. For smooth bounded kernels, we may use a theory of pseudo differential operators (e.g. Kumano-go [22], Komatsu [20]), while for measurable kernels, there exists a fundamental solution (see [21]). Also, we refer to [17] and Barlow, Bass, Chen, and Kassmann [1]. Recently, in Dyda and Kassmann [13], assumptions of kernels have been extended in some geometrical sense. If we focus on non-divergence case, we refer to [6].

As mentioned before, the following three sections 2, 3, and 4 are dedicated to the proof of the part (II) of the main theorem 1.1. More precisely, in Section 2, we introduce some definitions and few important lemmas. After that, we present and prove the main proposition 3.1 in Section 3. Finally, the proof of the part (II) of

Theorem 1.1 ends in Section 4. At the end of this paper, Appendix contains the proofs of the part (I) of Theorem 1.1 and Theorem 1.2.

2. Preliminaries and lemmas

From now on, we fix a parameter set $\{\alpha, \zeta, \omega, \Lambda\}$, which appears in Definition 1.1 (for the case $\alpha \geq 1$, $\{\alpha, \zeta, \omega, \Lambda, \nu, s_0, \tau\}$). Also, suppose that K satisfies the *weak-(∗)-kernel* condition in Definition 1.2 on the parameter set together with the smoothness assumption (10). For the case $\alpha < 1$, we define and fix a constant γ such that $0 < \gamma < (\alpha - \omega)$ while, for the case $\alpha \geq 1$, we take γ to be $\max\{(\alpha - N), 0\} < \gamma < (\alpha - (\omega + \nu)))$.

Before considering a general $\zeta \in (0, \infty]$, we will prove first the conclusion of the part (II) of Theorem 1.1 for a fixed $\zeta = \zeta_0$ where

$$(14) \qquad \zeta_0 := \max\left\{\left(\frac{8}{V_N}\right)^{1/N}, 2 \cdot (11)^{1/\gamma}\right\}$$

(V_N is the volume of the unit ball in $\mathbb{R}^N$). This definition of ζ_0 will help us to obtain enough regularization directly so that the proof becomes more straightforward. Once we prove the part (II) of Theorem 1.1 with $\zeta = \zeta_0$, a general proof for any value $\zeta \in (0, \infty]$ will follow a scaling argument. Indeed, the case $\zeta > \zeta_0$ is included in the case $\zeta = \zeta_0$ because $\{|x - y| \leq \zeta_0\} \subset \{|x - y| \leq \zeta\}$. On the other hand, for the case $\zeta < \zeta_0$, we define a scaling: $w^\epsilon(t, x) = w(\epsilon^\alpha t, \epsilon x)$ and $K^\epsilon(t, x, y) = K(\epsilon^\alpha t, \epsilon x, \epsilon y)$. Thus, if w satisfies (1) on $[0, T]$ for a kernel K with $\zeta < \zeta_0$, then w^ϵ is a solution on $[0, T/\epsilon^\alpha]$ for the kernel K^ϵ with a new $\zeta = \zeta_0$ once we pick up ϵ by $\epsilon = \zeta/\zeta_0$. Then we can apply the part (II) of Theorem 1.1 for w^ϵ and the same result for w follows.

In this paper, we denote Sobolev spaces by $W^{k,p}$ and $H^k := W^{k,2}$ for integers $k \geq 0$ and for $p \in [1, \infty]$ in the usual way. In addition, the symbol $\mathcal{S}$ is used to represent the Schwartz space in $\mathbb{R}^N$.

Definition 2.1. We say that a function f lies in $C^k(\mathbb{R}^d)$ for an integer $k \geq 0$ if f is k-times differentiable in $\mathbb{R}^d$ and all derivatives up to k order are continuous, while f lies in $C^k(\overline{\mathbb{R}^d})$ if $f \in C^k(\mathbb{R}^d)$ and if $\nabla^l f$ are bounded for all integer l such that $0 \leq l \leq k$. In other words, $C^k(\overline{\mathbb{R}^d}) = C^k(\mathbb{R}^d) \cap W^{k,\infty}(\mathbb{R}^d)$.

Definition 2.2. We say that a bounded function f lies in $C^\beta(\mathbb{R}^d)$ for $0 < \beta < 1$ if $\sup_{x,y} |f(x) - f(y)|/|x - y|^\beta$ is finite and we define the semi-norm $[f]_{C^\beta} := \sup_{x,y} |f(x) - f(y)|/|x - y|^\beta$ and the norm $\|f\|_{C^\beta} := \|f\|_{L^\infty} + [f]_{C^\beta}$. We also define the space $C^{k,\beta}(\mathbb{R}^d)$ by the norm $\|f\|_{C^{k,\beta}} := \|f\|_{W^{k,\infty}} + [\nabla^k f]_{C^\beta}$

It will be shown in Lemma 2.2 that the operator $T_t^K(f)$ is well-defined pointwise for $f \in (C^2 \cap L^1)(\mathbb{R}^N)$. Moreover the operator can be extended to more general spaces. For example, if f is locally integrable and $\int \frac{|f|}{1+|x|^{N+\alpha-\omega}} dx < \infty$, then we can define $T_t^K(f)$ as an element of $\mathcal{S}'$ where $\mathcal{S}'$ is the dual of Schwartz space $\mathcal{S}$ (see also Silvestre [26]). We will make use of the following Lemma 2.1, which says that $T_t^K(|\cdot|^\gamma)$ is not only an element of $\mathcal{S}'$ but also a locally integrable function with a desirable estimate. This fact will be used to obtain the concentration condition (Lemma 3.3) for the evolution of $\mathcal{U}_r$, which will be introduced in Definition 2.3.

Lemma 2.1. *We have an estimate*

$$\left| T_t^K \left(|\cdot|^\gamma \right)(x) \right| \leq \begin{cases} C \cdot |x|^{\gamma-\alpha} \cdot (1 + |x|^\omega), & \text{if } 0 < \alpha < 1, \\ C \cdot |x|^{\gamma-\alpha} \cdot (1 + |x|^{\nu+\omega}), & \text{if } 1 \leq \alpha < 2. \end{cases}$$

Remark 2.1. Recall that γ is a fixed constant such that $0 < \gamma < \alpha - \omega$ (for $\alpha \geq 1$, $\max\{(\alpha - N), 0\} < \gamma < (\alpha - (\omega + \nu)))$.

Proof.

$$\int_{\mathbb{R}^N} \left[|x|^\gamma - |y|^\gamma \right] K(t,x,y) dy = \int_{\mathbb{R}^N} \left[|x|^\gamma - |x+z|^\gamma \right] K(t,x,x+z) dz$$

$$= \int_{\mathbb{R}^N} \frac{|x|^\gamma - |x+z|^\gamma}{|z|^{N+\alpha}} k(t,x,z) dz$$

$$= |x|^\gamma \int_{\mathbb{R}^N} \frac{1 - |\frac{x}{|x|} + \frac{z}{|x|}|^\gamma}{|z|^{N+\alpha}} k(t,x,z) dz.$$

Then we use the change of variables $\frac{z}{|x|} = \bar{z}$ and the polar coordinate to get

$$= |x|^{\gamma-\alpha} \int_{\mathbb{R}^N} \frac{1 - |\frac{x}{|x|} + \bar{z}|^\gamma}{|\bar{z}|^{N+\alpha}} k(t,x,|x|\bar{z}) d\bar{z}$$

$$= |x|^{\gamma-\alpha} \int_{S^{N-1}} \int_0^\infty \frac{1 - |\frac{x}{|x|} + s\sigma|^\gamma}{s^{1+\alpha}} k(t,x,|x|s\sigma) ds d\sigma$$

$$= |x|^{\gamma-\alpha} \left(\int_{S^{N-1}} \int_0^{1/2} \cdots ds d\sigma + \int_{S^{N-1}} \int_{1/2}^\infty \cdots ds d\sigma \right) = |x|^{\gamma-\alpha} \Big((I) + (II) \Big).$$

From the condition $\gamma + \omega < \alpha$, we have

$$\left| (II) \right| \leq \Lambda \int_{S^{N-1}} \int_{1/2}^\infty \frac{(2 + s^\gamma)(1 + |x|^\omega s^\omega)}{s^{1+\alpha}} ds d\sigma \leq C(1 + |x|^\omega).$$

On the other hand, from Taylor expansion $1 - |\frac{x}{|x|} + s\sigma|^\gamma = -\gamma(\frac{x}{|x|} \cdot \sigma)s + R_\sigma(s)$ with an error estimate $|R_\sigma(s)| \leq Cs^2$ for $s \in [0, 1/2]$ and $\sigma \in S^{N-1}$, we have

$$\left| (I) \right| \leq \left| \int_{S^{N-1}} \int_0^{1/2} \gamma\left(\frac{x}{|x|} \cdot \sigma\right) \frac{s}{s^{1+\alpha}} k(t,x,|x|s\sigma) ds d\sigma \right|$$

$$+ \Lambda \int_{S^{N-1}} \int_0^{1/2} \frac{|R_\sigma(s)|(1 + |x|^\omega s^\omega)}{s^{1+\alpha}} ds d\sigma$$

$$\leq C \int_0^{1/2} \frac{1}{s^\alpha} \left| \int_{S^{N-1}} k(t,x,|x|s\sigma) \sigma d\sigma \right| ds + C \cdot \frac{1 + |x|^\omega}{2 - \alpha} \cdot \Lambda$$

$$= C \cdot (III) + C \cdot (1 + |x|^\omega).$$

For the case $\alpha < 1$, (III) is bounded above by $C \cdot (1 + |x|^\omega)$.
If $\alpha \geq 1$, we can use the condition (6), which is obtained from (5) and (3), together with $\nu > (\alpha - 1)$:

$$(III) \leq \int_0^{1/2} \frac{1}{s^\alpha} \bar{\tau} |x|^\nu s^\nu (1 + |x|^\omega s^\omega) ds \leq \bar{\tau} |x|^\nu \cdot (1 + |x|^\omega) \int_0^{1/2} \frac{1}{s^{\alpha-\nu}} ds$$

$$\leq C \cdot (1 + |x|^\omega) \cdot |x|^\nu.$$

$\square$

We give, in the following lemma, some properties of the integral operator T_t^K and the related evolution equation (1).

Lemma 2.2. *For any $f \in C^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $T_t^K(f)$ is well-defined pointwise. Moreover, the following properties hold:*
(I). Duality of T:

$$(15) \qquad \int f(x) T_t^K(g)(x) dx = \int T_t^K(f)(x) g(x) dx,$$

for $f \in C^2(\overline{\mathbb{R}^N}) \cap L^1(\mathbb{R}^N)$ and either $g \in C^2(\overline{\mathbb{R}^N}) \cap L^1(\mathbb{R}^N)$ or $g(x) = |x|^\gamma$.
(II). Mean zero of T:

$$\int T_t^K(f)(x) dx = 0,$$

for $f \in C^2(\overline{\mathbb{R}^N}) \cap L^1(\mathbb{R}^N)$.

Proof. Let $f \in C^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. Then, we have

$$T_t^K(f)(x) = \int (f(x) - f(y)) K(t,x,y) dy = \int \frac{f(x) - f(x+z)}{|z|^{N+\alpha}} k(t,x,z) dz$$

$$= \int_0^1 \int_{S^{N-1}} \frac{(\nabla f)(x) \cdot (r\sigma) + R_f(x,r\sigma)}{r^{1+\alpha}} k(t,x,r\sigma) d\sigma dr$$

$$+ \int_{|z| \geq 1} \frac{f(x) - f(x+z)}{z^{N+\alpha}} k(t,x,z) dz = (a) + (b).$$

where we used the Taylor expansion of f in the first integral.
For (b), we use the upper bound of (3):

$$|(b)| \leq \Lambda \int_{|z| \geq 1} \left(|f(x)| + |f(x+z)| \right) \frac{1 + |z|^\omega}{|z|^{N+\alpha}} dz$$

$$\leq C|f(x)| \int_{|z| \geq 1} \frac{1}{|z|^{N+\alpha-\omega}} dz + C \int_{|z| \geq 1} |f(x+z)| dz$$

$$\leq C|f(x)| + \|f\|_{L^1}.$$

For (a), if $\alpha < 1$, we use $|R_f(x,r\sigma)| \leq C \cdot \|\nabla^2 f\|_{L^\infty(B_x(1))} \cdot r^2$ from the Taylor error estimate where $B_x(r)$ is the ball of radius r centered at x:

$$|(a)| \leq C|\nabla f(x)| \cdot \int_0^1 \frac{r}{r^{1+\alpha}} + C\|\nabla^2 f\|_{L^\infty(B_x(1))} \cdot \int_0^1 \frac{r^2}{r^{1+\alpha}}$$

$$\leq C\left(|\nabla f(x)| + \|\nabla^2 f\|_{L^\infty(B_x(1))} \right).$$

If $\alpha \geq 1$, we use the condition (6) with the assumption $(\alpha - 1) < \nu$:

$$(a) \leq C|\nabla f(x)| \cdot \int_0^1 \frac{1}{r^\alpha} \left| \int_{S^{N-1}} k(t,x,r\sigma) \sigma d\sigma \right| dr + C\|\nabla^2 f\|_{L^\infty(B_x(1))}$$

$$\leq C|\nabla f(x)| \cdot \int_0^1 \frac{1}{r^\alpha} r^\nu dr + C\|\nabla^2 f\|_{L^\infty(B_x(1))}$$

$$\leq C\left(|\nabla f(x)| + \|\nabla^2 f\|_{L^\infty(B_x(1))} \right).$$

Now we can easily verify that $T_t^K(f)$ is well-defined pointwise.

Note that if $f \in C^2(\overline{\mathbb{R}^N}) \cap L^1(\mathbb{R}^N)$, then the above argument implies $T_t^K(f) \in L^\infty$. Then, the proof of (I) follows the symmetry in x, y of K. Indeed, if $f, g \in C^2(\overline{\mathbb{R}^N}) \cap L^1(\mathbb{R}^N)$, then we have

$$
\begin{aligned}
\int f(x) T_t^K(g)(x) dx &= \int \int f(x)(g(x) - g(y)) K(t, x, y) dy dx \\
&= \frac{1}{2} \int \int (f(x) - f(y))(g(x) - g(y)) K(t, x, y) dy dx \\
&= \int \int (f(x) - f(y)) g(x) K(t, x, y) dy dx \\
&= \int T_t^K(f)(x) g(x) dx.
\end{aligned}
$$

(16)

In addition, for the case $f \in C^2(\overline{\mathbb{R}^N}) \cap L^1(\mathbb{R}^N)$ with $g(x) = |x|^\gamma$, then the integral $\int |f(x) T_t^K(g)(x)| dx$ is bounded due to the assumption $f \in L^\infty \cap L^1$ with Lemma 2.1. Indeed, Lemma 2.1 implies that $T_t^K(|\cdot|^\gamma)$ is integrable in the unit ball containing the origin and is bounded outside of the ball. Then, together with $f \in L^\infty \cap L^1$, we obtain $f \cdot T_t^K(|\cdot|^\gamma) \in L^1$. Thus all equalities of (16) can be justified via a limit argument.

To prove (II), we take $g \in C_c^\infty$ such that $g = 1$ in $B(1)$ and $supp(g) \subset B(2)$ and define g_n by $g_n(\cdot) := g(\cdot/n)$. Then, thanks to the property (I) with g_n, the conclusion follows by taking a limit $n \to \infty$. $\square$

In the following lemma, we present a maximum principle for solutions of (1).

Lemma 2.3. *Suppose that $w \in L_t^\infty(L_x^\infty \cap L_x^1)$ is a smooth solution of (1). Then, the following properties hold:*
(I). For any C^2 convex function $\eta : \mathbb{R} \to \mathbb{R}$, we have

$$
\text{(17)} \qquad\qquad \partial_t(\eta(w)) + T_t^K(\eta(w)) \le 0.
$$

(II). L^p−norm is non-increasing for $1 \le p \le \infty$:

$$
\text{(18)} \qquad\qquad \|w(t, \cdot)\|_{L^p(\mathbb{R}^N)} \le \|w(s, \cdot)\|_{L^p(\mathbb{R}^N)} \text{ for any } s < t.
$$

Remark 2.2. Also we assume that the solutions are smooth. However the estimate of the result does not depend on this smoothness.

Proof. To prove (I), we multiply $\eta'(w)$ to the equation (1) to get $\partial_t(\eta(w)) + \eta'(w) T_t^K(w) = 0$. Then it is enough to show $\eta'(w) T_t^K(w) - T_t^K(\eta(w)) \ge 0$. Using the integral representation of T_t^K, we have

$$
\begin{aligned}
\eta'(w(x)) &\Big(T_t^K(w) \Big)(x) - \Big(T_t^K(\eta(w)) \Big)(x) = \\
&\int (w(x) - w(y)) K(x, y) \eta'(w(x)) dy - \int (\eta(w(x)) - \eta(w(y))) K(x, y) dy \\
&= \int \Big(\eta'(w(x))(w(x) - w(y)) - (\eta(w(x)) - \eta(w(y))) \Big) K(x, y) dy \ge 0
\end{aligned}
$$

because $\eta'(a)(a - b) - (\eta(a) - \eta(b)) \ge 0$ from convexity of η.

To prove (II), for $p \geq 2$, we use (17) with putting $\eta(\cdot) = |\cdot|^p$ and taking an integral in x variable in order to use (II) of Lemma 2.2. For $p < 2$, we need to regularize $\eta(\cdot) = |\cdot|^p$ first. $\qquad\qquad\square$

Now we adopt the notion of the class $\mathcal{U}_r$ of test functions following the paper [18]. Let $A \geq 1$ be a constant which will be chosen later.

Definition 2.3. We say that a measurable function $\varphi(\cdot)$ on $\mathbb{R}^N$ lies in $\mathcal{U}_r$ for some $r \in (0, \infty)$ if φ satisfies the following four conditions:

$$\int_{\mathbb{R}^N} \varphi(x)dx = 0 \qquad \text{the mean zero-condition,}$$

$$\int_{\mathbb{R}^N} |\varphi(x)||x - x_0|^\gamma dx \leq r^\gamma \quad \text{for some } x_0 \in \mathbb{R}^N \quad \text{the concentration-condition,}$$

$$\|\varphi\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A}{r^N} \qquad \text{the } L^\infty\text{-condition, and}$$

$$\|\varphi\|_{L^1(\mathbb{R}^N)} \leq 1 \qquad \text{the } L^1\text{-condition.}$$

In addition, we say that φ lies in $a\mathcal{U}_r$ for some $a > 0$ when $(1/a)\varphi \in \mathcal{U}_r$. We call x_0 a center of φ.

The following lemma connects between C^β space and $\mathcal{U}_r$, which tells us that $r^{-\beta}\mathcal{U}_r$ plays a similar role of the dual space of C^β.

Lemma 2.4. *Let β be any constant such that $0 < \beta \leq \gamma$.*
(I) Then we have

$$\left| \int_{\mathbb{R}^N} w(x)\varphi(x)dx \right| \leq r^\beta [w]_{C^\beta(\mathbb{R}^N)}$$

for any $w \in C^\beta(\mathbb{R}^N)$, for any $0 < r < \infty$, and for any $\varphi \in \mathcal{U}_r$.

(II) Conversely, we have a constant C such that
if a bounded function w satisfies $\displaystyle\sup_{\varphi \in \mathcal{S} \cap \mathcal{U}_r, 0 < r \leq 1} r^{-\beta} \left| \int_{\mathbb{R}^3} w(x)\varphi(x)dx \right| < \infty$,
then $w \in C^\beta$ and

$$(19) \qquad \|w\|_{C^\beta(\mathbb{R}^N)} \leq C \cdot \left(\|w\|_{L^\infty} + \sup_{\varphi \in \mathcal{S} \cap \mathcal{U}_r, 0 < r \leq 1} r^{-\beta} \left| \int_{\mathbb{R}^3} w(x)\varphi(x)dx \right| \right).$$

Proof. For the part *(I)*, let x_0 be a center of φ. Then, from the mean zero property,

$$\left| \int_{\mathbb{R}^N} w(x)\varphi(x)dx \right| \leq \left| \int_{\mathbb{R}^N} \left(w(x) - w(x_0) \right)\varphi(x)dx \right|$$

$$\leq [w]_{C^\beta(\mathbb{R}^N)} \int_{\mathbb{R}^N} |x - x_0|^\beta |\varphi(x)|dx$$

$$\leq [w]_{C^\beta(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} |x - x_0|^\gamma |\varphi(x)|dx \right)^{\beta/\gamma} \left(\int_{\mathbb{R}^N} |\varphi(x)|dx \right)^{(\gamma-\beta)/\gamma} \leq r^\beta [w]_{C^\beta(\mathbb{R}^N)}.$$

For the part *(II)*, we recall Littlewood-Paley projections Δ_j, which is defined by $\Delta_j(w) = w * \Psi_{2^{-j}}$ where $\Psi_t(x) = t^{-N}\Psi(x/t)$ and $\hat{\Psi}(\xi) = \eta(\xi) - \eta(2\xi)$ with $\eta \in C_0^\infty, 0 \leq \eta(\xi) \leq 1, \eta = 1$ for $|\xi| \leq 1$ and $\eta = 0$ for $|\xi| \geq 2$. We use the characterization of C^β in terms of Littlewood-Paley projections (see Stein [27]). Indeed, if a bounded function w in $\mathbb{R}^N$ satisfies

$$\sup_{j=1,2,3,\cdots} 2^{\beta j} \|\Delta_j(w)\|_{L^\infty(\mathbb{R}^N)} < \infty$$

then w lies in $C^\beta(\mathbb{R}^N)$ and it has the estimate

$$\|w\|_{C^\beta(\mathbb{R}^N)} \leq C_1(\|w\|_{L^\infty(\mathbb{R}^N)} + \sup_{j=1,2,3,\cdots} 2^{\beta j}\|\Delta_j(w)\|_{L^\infty(\mathbb{R}^N)})$$

where C_1 depends only on β, N and the choice of Ψ. In order to show (19), it is enough to find $0 < a < \infty$ such that $\Psi_{2^{-j}} \in a\mathcal{U}_{2^{-j}}$ for all $j \geq 1$ because $\Delta_j(w)(x) = \int_{\mathbb{R}^3} w(y)\Psi_{2^{-j}}(x - y)dy$ and $\mathcal{U}_r$ is translation invariant.

It is clear that Ψ is a Schwartz function from the fact $\eta \in C_0^\infty$. Thus we can take $a := \|\Psi\|_{L^\infty(\mathbb{R}^3)} + \|\Psi\|_{L^1(\mathbb{R}^3)} + \int_{\mathbb{R}^N} |\Psi(x)||x|^\gamma dx < \infty$. Then, for any $r > 0$, we have

$$\int (1/a)\Psi_r = (1/a)\int \Psi = (1/a)\hat{\Psi}(0) = 0,$$

$$\|(1/a)\Psi_r\|_{L^\infty} \leq (1/a)r^{-N}\|\Psi\|_{L^\infty} \leq r^{-N} \leq \frac{A}{r^N},$$

$$\|(1/a)\Psi_r\|_{L^1} \leq (1/a)\|\Psi\|_{L^1} \leq 1, \text{ and}$$

$$\int_{\mathbb{R}^N} |(1/a)\Psi_r(x)||x|^\gamma dx = (1/a)r^\gamma \cdot \int_{\mathbb{R}^N} |\Psi(x)||x|^\gamma dx \leq r^\gamma.$$

Thus (19) follows with $C = C_1 \cdot \max\{1, a\}$.

$\square$

We define the backward kernel $K^{(\overline{T})}$ corresponding to any finite time $\overline{T} < T$ and to the kernel K by

(20) $$K^{(\overline{T})}(s, x, y) = K(\overline{T} - s, x, y).$$

Then it is easy to see $T_t^K = T_{\overline{T}-t}^{K^{(\overline{T})}}$ and they share the *weak-(∗)-kernel* condition with the same parameter set.

Lemma 2.5. *Let* $w, \varphi \in L^\infty(0, \overline{T}; (L^1 \cap L^\infty)(\mathbb{R}^N))$ *be two smooth solutions of* (1) *with* $\overline{T} < \infty$ *for each smooth initial data* $w_0, \varphi_0 \in (L^1 \cap L^\infty)(\mathbb{R}^N)$ *and for each associated kernels* K *and* $K^{(\overline{T})}$, *respectively. In addition, we assume* $\varphi_0 \in \mathcal{U}_r \cap \mathcal{S}$ *for some* $r \in (0, 1]$. *Then, we have*

$$\int_{\mathbb{R}^3} w_0(x)\varphi(\overline{T}, x)dx = \int_{\mathbb{R}^3} w(\overline{T}, x)\varphi_0(x)dx.$$

Proof. Let $t \in [0, \overline{T}]$. Then, we have

$$\frac{d}{dt}\int_{\mathbb{R}^3} w(t, x)\varphi(\overline{T} - t, x)dx$$

$$= \int_{\mathbb{R}^3} (\partial_t w)(t, x)\varphi(\overline{T} - t, x)dx - \int_{\mathbb{R}^3} w(t, x)(\partial_t \varphi)(\overline{T} - t, x)dx$$

$$= -\int_{\mathbb{R}^3} T_t^K(w(t, \cdot))(x)\varphi(\overline{T} - t, x)dx + \int_{\mathbb{R}^3} w(t, x)T_{\overline{T}-t}^{K^{(\overline{T})}}(\varphi(\overline{T} - t, \cdot))(x)dx.$$

Then, we use Lemma 2.2 and the fact $T_t^K = T_{\overline{T}-t}^{K^{(\overline{T})}}$ to get

$$= -\int_{\mathbb{R}^3} T_t^K(w(t, \cdot))(x)\varphi(\overline{T} - t, x)dx + \int_{\mathbb{R}^3} T_{\overline{T}-t}^{K^{(\overline{T})}}(w(t, \cdot))(x)\varphi(\overline{T} - t, x)dx$$

$$= -\int_{\mathbb{R}^3} T_t^K(w(t, \cdot))(x)\varphi(\overline{T} - t, x)dx + \int_{\mathbb{R}^3} T_t^K(w(t, \cdot))(x)\varphi(\overline{T} - t, x)dx = 0.$$

As a result, we conclude that $\int_{\mathbb{R}^3} w(t,x)\varphi(\overline{T}-t,x)dx$ is constant in t. Then put $t := 0$ and $t := \overline{T}$.

$\square$

3. The main proposition and its proof

We are ready to present the main proposition about the evolution of test functions in a short time interval, whose length is proportional to r^α. Roughly speaking, if $\varphi_0 \in \mathcal{U}_r$, then there exist $z = z(r,s)$ and β such that $\varphi(s) \in \left(\frac{r}{z}\right)^\beta \mathcal{U}_z$ for $s \in [0, \delta r^\alpha]$.

Proposition 3.1. *There exist constants $A \geq 1$, $\delta > 0$, $L > 0$ and $\beta > 0$ with the following property:*

Let $0 < r \leq 1$ and $\varphi_0 \in \mathcal{U}_r \cap \mathcal{S}$. Then, there exist a smooth solution $\varphi \in L^\infty(0,T;(L^1 \cap L^\infty)(\mathbb{R}^N))$ of (1) with the initial condition $\varphi(0) = \varphi_0$. Also, for any $s \in [0, \min\{\delta r^\alpha, T\}]$, we have

$$\varphi(s) \in \left(\frac{r}{z(r,s)}\right)^\beta \mathcal{U}_{z(r,s)} \tag{21}$$

where $z(r,s)$ is defined by $z(r,s) = r(1 + L\frac{s}{r^\alpha})$.
Moreover, if $r = 1$, then

$$\varphi(s) \in (1 + Ls)\mathcal{U}_1 \quad \text{for any } s \in [0,T). \tag{22}$$

Proof. Let $\varphi_0 \in \mathcal{U}_r \cap \mathcal{S}$ for some $0 < r \leq 1$. Then there exists a weak solution φ corresponding to the initial data φ_0 (this can be proved by following [21]. Or refer to the approximation scheme in [4]). Moreover this solution is smooth, and it lies in $L_t^\infty(H_x^b)$ for every integer $b \geq 0$ due to the smoothness assumption (10) of k (it can be proved by using a standard energy argument).

First we state the following elementary inequalities without proof.

Lemma 3.2. *(I). $(1-x) \leq \frac{1}{1+x}$ for $x \geq 0$.*
(II). $(1 + \frac{\eta}{2}x) \leq (1+x)^\eta$ for any $0 \leq x \leq 1$ if $0 \leq \eta \leq 1$.
(III). $(1+x)^\eta \leq (1+\eta x)$ for any $x \geq 0$ if $0 \leq \eta \leq 1$.
(IV). $(1+x)^\eta \leq (1+2\eta x)$ for any $0 < x < C_\eta$ if $\eta \geq 1$.

In order to obtain (21), we need to verify the mean zero, the concentration, the L^∞, and the L^1 conditions. First the mean-zero condition is easily verified in STEP 1. Second, we derive some estimates for remained three other conditions in STEP 2-4. Then, in STEP 5, we combine all the estimates we obtained in STEP 2-4 to finish the proof. Without loss of generality, we can assume that a center of φ_0 is the origin (i.e. $x_0 = 0$).

STEP 1. Mean zero-condition.
From (II) of Lemma 2.2, we have, for any $t \in (0,T)$,

$$\frac{d}{dt}\int_{\mathbb{R}^N} \varphi(t,x)dx = \int_{\mathbb{R}^N} \left(\frac{\partial}{\partial t}\varphi\right)(t,x)dx$$

$$= -\int_{\mathbb{R}^N} T_t^K(\varphi(t,\cdot))(x)dx = 0.$$

STEP 2. Concentration-condition.

Lemma 3.3. *There exists a constant $C_{conc} > 0$ such that, for any $s \in (0, T)$, we have*

$$(23) \qquad \int_{\mathbb{R}^N} |\varphi(s, x)||x|^\gamma dx \le r^\gamma (1 + C_{conc} A^{\frac{\alpha - \gamma}{N}} \frac{s}{r^\alpha})$$

where C_{conc} does not depend on A as long as $A \ge 1$.

Remark 3.1. This lemma says that test functions lose their concentration with certain rate as time goes on. In Step 5, it will be shown that the rate can be absorbed into the regularization effect from the L^1 and the L^∞ conditions.

Proof.

$$\frac{d}{ds} \int_{\mathbb{R}^N} |\varphi(s, x)||x|^\gamma dx = \int_{\mathbb{R}^N} \frac{\partial}{\partial s} \Big(|\varphi(s, x)| \Big) |x|^\gamma dx$$

$$\le \int_{\mathbb{R}^N} -T_s \Big(|\varphi(s, x)| \Big) \cdot |x|^\gamma dx$$

$$= \int_{\mathbb{R}^N} -T_s \Big(|x|^\gamma \Big) \cdot |\varphi(s, x)| dx$$

$$\le \int_{\mathbb{R}^N} \Big| T_s \Big(|x|^\gamma \Big) \Big| \cdot |\varphi(s, x)| dx = (I)$$

where we used Lemma 2.3 and Lemma 2.2.
First, consider the case $\alpha < 1$. Then, thanks to Lemma 2.1, we have

$$(I) \le C \int_{\mathbb{R}^N} |x|^{\gamma - \alpha} \cdot (1 + |x|^\omega) \cdot |\varphi(s, x)| dx$$

$$= C \Big(\int_{B(A^{-1/N} r)} |x|^{\gamma - \alpha} \cdot (1 + |x|^\omega) \cdot |\varphi(s, x)| dx$$

$$+ \int_{B(A^{-1/N} r)^C} |x|^{\gamma - \alpha} \cdot |\varphi(s, x)| dx$$

$$+ \int_{B(A^{-1/N} r)^C} |x|^{\gamma - \alpha + \omega} \cdot |\varphi(s, x)| dx \Big).$$

From the condition $\gamma < (\alpha - \omega)$, we have decreasing of the functions $|\cdot|^{\gamma - \alpha}$ and $|\cdot|^{\gamma - \alpha + \omega}$. Also, note that L^∞ and L^1 norms are decreasing and $A^{-1/N} \cdot r \le 1$ from $A \ge 1$ and $r \le 1$. Thus we have

$$\le C \Big((A^{-1/N} r)^{\gamma - \alpha + N} \cdot (1 + (A^{-1/N} r)^\omega) \cdot \|\varphi(s)\|_{L^\infty}$$

$$+ (A^{-1/N} r)^{\gamma - \alpha} \|\varphi(s)\|_{L^1} + (A^{-1/N} r)^{\gamma - \alpha + \omega} \|\varphi(s)\|_{L^1} \Big)$$

$$\le C A^{\frac{\alpha - \gamma}{N}} r^{\gamma - \alpha}.$$

Likewise, for the case $\alpha \ge 1$, Lemma 2.1 with $\gamma < \alpha - (\nu + \omega)$ gives us the same conclusion. Then, we have (23) thanks to the initial condition $\int_{\mathbb{R}^N} |\varphi(0, x)||x|^\gamma dx \le r^\gamma$. $\qquad \square$

STEP 3. L^∞-condition.

Lemma 3.4. *There exist two constants $\delta_{L^\infty} > 0$ and $C_{L^\infty} > 0$ such that, for any $s \in [0, \min\{\delta_{L^\infty} r^\alpha, T\}]$, we have*

$$(24) \qquad \|\varphi(s, \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A}{r^N}(1 - C_{L^\infty} A^{\frac{\alpha}{N}} \frac{s}{r^\alpha})$$

where C_{L^∞} does not depend on A as long as $A \geq 1$.

Remark 3.2. This lemma is proved by using the lower bound of (3), which gives us some regularization effect. We follow a similar argument of Theorem 4.1 in the paper Córdoba and Córdoba [11], which showed a L^∞ decay for smooth solutions of the 2D surface QG equation.

Proof. First, we define $M(t) := \|\varphi(t, \cdot)\|_{L^\infty}$. We claim that there exist $\delta_1 > 0$ and $C_1 > 0$ such that for any $t \in [0, \delta_1 r^\alpha]$ satisfying $M(t) \geq \frac{1}{2}\frac{A}{r^N}$, we have

$$(25) \qquad M(t) \leq \frac{A}{r^N}(1 - C_1 A^{\frac{\alpha}{N}} \frac{t}{r^\alpha}).$$

To prove the above claim (25), first pick any $t \in (0, T)$ such that

$$M(t) \geq \frac{1}{2} \cdot \frac{A}{r^N}.$$

Then we know $M(t^*) \geq \frac{1}{2}\frac{A}{r^N}$ for all $t^* < t$ from Lemma 2.3. It can be easily proved that there exists a point x_t such that $|\varphi(t, x_t)| = M(t)$. Indeed, because our kernel lies in $C^\infty_{t,x,y}\left([0,T] \times \overline{\mathbb{R}^N} \times \overline{\mathbb{R}^N}\right)$ with $\varphi_0 \in \mathcal{S}$, we can show $\varphi \in L^\infty_t H^d$ for every integer $d \geq 0$ by standard energy estimates. In particular, $\varphi(t, \cdot) \in H^b$ for some integer $b > (N/2)$ for every time. Then, $\varphi(t, \cdot)$ vanishes at the infinity thanks to a Fourier transform argument. Since $\varphi(t, \cdot)$ is continuous, there exists a maximum (or minimum) point.

Then, for almost every time $t \in (0, T)$, there exist a point $\tilde{x}_t$ such that $|\varphi(t, \tilde{x}_t)| = M(t)$ with the following inequality:

$$\frac{d}{dt}M(t) \leq \begin{cases} \left(\frac{\partial \varphi}{\partial t}\right)(t, \tilde{x}_t) & \text{if } \varphi(t, \tilde{x}_t) = M(t), \\ -\left(\frac{\partial \varphi}{\partial t}\right)(t, \tilde{x}_t) & \text{if } \varphi(t, \tilde{x}_t) = -M(t) \end{cases}$$

(this can be proved by following the argument of [11]).

We assume the first case $\varphi(t, \tilde{x}_t) = M(t) > 0$ (the other one can be dealt in similar fashion). Then

$$\frac{d}{dt}M(t) \leq \left(\frac{\partial \varphi}{\partial t}\right)(t, \tilde{x}_t) \leq -T_t^K(\varphi(t, \cdot))(x)$$

$$(26) \qquad = -\int_{\mathbb{R}^N} \left(\varphi(t, \tilde{x}_t) - \varphi(t, y)\right)K(t, \tilde{x}_t, y)dy$$

$$\leq -\Lambda^{-1}\int_{|\tilde{x}_t - y| \leq \zeta} \frac{\varphi(t, \tilde{x}_t) - \varphi(t, y)}{|\tilde{x}_t - y|^{N+\alpha}}dy = -\Lambda^{-1} \cdot (*).$$

We used the fact $\varphi(t, \tilde{x}_t) - \varphi(t, y) \geq 0$ with the lower bound of the kernel (3).

Let R be any number between 0 and ζ, which will be chosen soon. We separate the ball $B_R(\tilde{x}_t)$ into two disjoint regions Ω_1 and Ω_2 by the following way:

$\big(\varphi(t, \tilde{x}_t) - \varphi(t, y)\big) > \frac{1}{2}\varphi(t, \tilde{x}_t)$ implies $y \in \Omega_1$. Otherwise, $y \in \Omega_2$. Then we have the following upper bound of measure of Ω_2:

$$|\Omega_2| = \frac{2}{M(t)}\int_{\Omega_2}\frac{M(t)}{2}dy \le \frac{2}{M(t)}\int_{\Omega_2}\varphi(t,y)dy \le \frac{2}{M(t)}\|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^N)} \le \frac{2}{M(t)}.$$

As a result, from $R < \zeta$, we have

$$(*) \ge \int_{|\tilde{x}_t - y| \le R}\frac{\varphi(t,\tilde{x}_t) - \varphi(t,y)}{|\tilde{x}_t - y|^{N+\alpha}}dy \ge \int_{\Omega_1}\frac{\varphi(t,\tilde{x}_t) - \varphi(t,y)}{|\tilde{x}_t - y|^{N+\alpha}}dy$$

$$\ge \frac{M(t)}{2R^{N+\alpha}}|\Omega_1| = \frac{M(t)}{2R^{N+\alpha}}(|B_R(\tilde{x}_t)| - |\Omega_2|) \ge \frac{M(t)}{2R^{N+\alpha}}(V_N R^N - \frac{2}{M(t)}).$$

Now we choose R by $V_N R^N = \frac{2}{M(t)} \cdot 2$. Then, it is clear that $R \le \zeta$ because $M(t) \ge \frac{1}{2}\frac{A}{r^N} \ge \frac{1}{2}\frac{1}{r^N} \ge \frac{1}{2}$ and $R = \left(\frac{4}{M(t)V_N}\right)^{1/N} \le \left(\frac{8}{V_N}\right)^{1/N} \le \zeta$ by (14). Coming back to (26), we have

$$\frac{d}{dt}M(t) \le -\Lambda^{-1}\frac{M(t)}{2R^{N+\alpha}}(V_N R^N - \frac{2}{M(t)}) = -\Lambda^{-1}\frac{M(t)}{2R^{N+\alpha}} \cdot \frac{2}{M(t)} = -\Lambda^{-1}\frac{1}{R^{N+\alpha}}$$

$$= -\Lambda^{-1}\Big(\frac{M(t)V_N}{4}\Big)^{(N+\alpha)/N} = -C \cdot M(t)^{1+\frac{\alpha}{N}}.$$

Solving this differential inequality, we obtain

$$M(t) \le M(0)\Big(1 + C \cdot M(0)^{\alpha/N} \cdot t\Big)^{-N/\alpha}$$

From the fact $\frac{1}{2}\frac{A}{r^N} \le M(t) \le M(0) \le \frac{A}{r^N}$, we have

$$\le \frac{A}{r^N}\Big(1 + C \cdot \Big(\frac{1}{2} \cdot \frac{A}{r^N}\Big)^{\alpha/N} \cdot t\Big)^{-N/\alpha}$$

For any $p > 0$, it is easy to see $(1 + x)^{-p} \le (1 - \frac{1}{2}px)$ for $0 \le x \le C_p$. Thus, we have

$$\le \frac{A}{r^N}\Big(1 - C_1 \cdot A^{\alpha/N} \cdot \frac{t}{r^\alpha}\Big)$$

as long as $t \le C_1^{-1}A^{-\alpha/N}r^\alpha$. By taking $\delta_1 := C_1^{-1}A^{-\alpha/N}$, we proved the claim (25) under the assumption $M(t) \ge \frac{1}{2}\frac{A}{r^N}$.

Thanks to (25), the whole case (24) can be achieved easily by taking $\delta_{L^\infty} := \frac{1}{2}\delta_1$ and $C_{L^\infty} := C_1$. Indeed, if $M(t) \le \frac{1}{2}\frac{A}{r^N}$, then we have

$$M(t) \le \frac{1}{2}\frac{A}{r^N} \le \frac{A}{r^N}(1 - C_1 A^{\frac{\alpha}{N}}\frac{t}{r^\alpha})$$

as long as $t \le \frac{1}{2}C_1^{-1}A^{-\alpha/N}r^\alpha = \frac{1}{2}\delta_1 r^\alpha$.

$\square$

STEP 4. L^1-condition.

Lemma 3.5. *There exist two constants $\delta_{L^1} > 0$ and $C_{L^1} > 0$ such that, for any $s \in [0, \min\{\delta_{L^1}r^\alpha, T\}]$, we have*

$$(27) \qquad\qquad \|\varphi(s,\cdot)\|_{L^1(\mathbb{R}^N)} \le (1 - C_{L^1} \cdot \frac{s}{r^\alpha})$$

where C_{L^1} does not depend on A as long as $A \geq 1$.

Remark 3.3. In this time, we obtain L^1 decay by using the lower bound of the kernel (3). In general, without the mean zero property, we do not expect L^1 decay (refer to [11]). However, with the mean zero property, we can manage certain amount of cancellation of the L^1-norm. This idea comes from the argument in [18] where L^1 decay for mean-zero solutions for the 2D-SQG equation in a periodic setting was obtained.

Proof. First, by using (23), we can find δ_2 such that $\int_{\mathbb{R}^N} |\varphi(s,x)||x|^\gamma dx \leq \frac{11}{10} r^\gamma$ for all $t \in [0, \delta_2 r^\alpha]$. i.e. we take δ_2 so small that $C_{conc} A^{\frac{\alpha-\gamma}{N}} \delta_2 \leq \frac{1}{10}$.

We claim that there exists a constant $C_2 > 0$ such that for any $t \in [0, \delta_2 r^\alpha]$ satisfying $\|\varphi(t)\|_{L^1(\mathbb{R}^N)} \geq \frac{9}{10}$, we have

$$(28) \qquad \|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^N)} \leq (1 - C_2 \cdot \frac{t}{r^\alpha}).$$

To prove (28), let $t \in [0, \delta_2 r^\alpha]$ satisfy $\|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^N)} \geq \frac{9}{10}$. For simplicity, we define $a := (11)^{1/\gamma}$. Then, from (14), we know

$$(29) \qquad\qquad\qquad 2a \leq \zeta$$

and the following estimates hold:
(30)
$$\int_{B(ar)} |\varphi(t,x)|dx \geq \frac{8}{10}, \int_{B(ar)} \varphi^+(t,x)dx \geq \frac{3}{10}, \text{ and } \int_{B(ar)} \varphi^-(t,x)dx \geq \frac{3}{10}.$$

where $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$.

Indeed, from the concentration condition, we obtain the following upper bound of L^1-norm outside of the ball $B(ar)$:

$$\int_{B(ar)^C} |\varphi(t,x)|dx = \int_{B(ar)^C} |\varphi(t,x)||x|^\gamma |x|^{-\gamma} dx \leq \frac{11}{10} r^\gamma (ar)^{-\gamma} = \frac{1}{10}.$$

Then, thanks to the mean-zero property, we get the following lower bounds of $\|\varphi\|_{L^1(B(ar))}$, $\|\varphi^+\|_{L^1(B(ar))}$ and $\|\varphi^-\|_{L^1(B(ar))}$:

$$\int_{B(ar)} |\varphi(t,x)|dx = \|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^N)} - \int_{B(ar)^C} |\varphi(t,x)|dx \geq \frac{9}{10} - \frac{1}{10} = \frac{8}{10},$$

$$\int_{B(ar)} \varphi^\pm(t,x)dx = \int_{\mathbb{R}^N} \varphi^\mp(t,x)dx - \int_{B(ar)^C} \varphi(t,x)^\pm dx$$

$$= \frac{1}{2}\|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^N)} - \int_{B(ar)^C} \varphi(t,x)^\pm dx$$

$$\geq \frac{1}{2}\|\varphi(t,\cdot)\|_{L^1(\mathbb{R}^N)} - \int_{B(ar)^C} |\varphi(t,x)|dx \geq \frac{9}{20} - \frac{1}{10} > \frac{3}{10}.$$

We denote symbols D_+^s, D_-^s and S^s by

$$D_\pm^s = \{x \in \mathbb{R}^N | \quad \pm \varphi(s,x) \geq 0\} \text{ and}$$
$$S^s = \{x \in \mathbb{R}^N | \quad \varphi(s,x) = 0\}.$$

Then, we have

$$\frac{d}{dt}\int_{\mathbb{R}^N}|\varphi(s,x)|dx = \int_{\mathbb{R}^N}\frac{\partial}{\partial t}|\varphi(s,x)|dx = \int_{\mathbb{R}^N}\Big(1_{D_+^s}(x)-1_{D_-^s}(x)\Big)(\partial_t\varphi)(s,x)dx$$

$$= -\int_{\mathbb{R}^N}\Big(1_{D_+^s}(x)-1_{D_-^s}(x)\Big)T_s^K(\varphi(s,\cdot))(x)dx$$

$$= -\int_{\mathbb{R}^N}\Big(1_{D_+^s}(x)-1_{D_-^s}(x)\Big)\int_{\mathbb{R}^N}[\varphi(s,x)-\varphi(s,y)]K(s,x,y)dydx$$

$$= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big[\Big(1_{D_+^s}(x)-1_{D_-^s}(x)\Big)-\Big(1_{D_+^s}(y)-1_{D_-^s}(y)\Big)\Big]\Big(\varphi(s,x)-\varphi(s,y)\Big)\cdot$$
$$K(s,x,y)dydx$$

$$= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(*)dydx$$

where we use the simplification $(*) :=$

$$\Big[\Big(1_{D_+^s}(x)-1_{D_-^s}(x)\Big)-\Big(1_{D_+^s}(y)-1_{D_-^s}(y)\Big)\Big]\Big(\varphi(s,x)-\varphi(s,y)\Big)\cdot K(s,x,y).$$

Then, we split the above integral into 9 components:

$$= -\frac{1}{2}\Big[\int_{D_+^s}\int_{D_+^s}(*)dydx + \int_{D_+^s}\int_{S^s}(*)dydx + \int_{D_+^s}\int_{D_-^s}(*)dydx$$

$$+ \int_{S^s}\int_{D_+^s}(*)dydx + \int_{S^s}\int_{S^s}(*)dydx + \int_{S^s}\int_{D_-^s}(*)dydx$$

$$+ \int_{D_-^s}\int_{D_+^s}(*)dydx + \int_{D_-^s}\int_{S^s}(*)dydx + \int_{D_-^s}\int_{D_-^s}(*)dydx\Big]$$

$$= -\frac{1}{2}\Big[(I)+(II)+\cdots+(IX)\Big]$$

We will prove the inequality: $\frac{1}{2}\Big[(I)+(II)+\cdots+(IX)\Big]\geq C_2\cdot r^{-\alpha}$, which will imply the claim (28) later. First, we observe that $(I)=(V)=(IX)=0$ by the definition of $(*)$.

Second, we have $(II)=(IV)$ by symmetry of the kernel. Indeed,

$$(II) = \int_{D_+^s}\int_{S^s}(*)dydx = \int_{D_+^s}\int_{S^s}\varphi(s,x)\cdot K(s,x,y)dydx$$

$$= \int_{S^s}\int_{D_+^s}\varphi(s,x)\cdot K(s,x,y)dxdy = \int_{S^s}\int_{D_+^s}\varphi(s,y)\cdot K(s,y,x)dydx$$

$$= \int_{S^s}\int_{D_+^s}\varphi(s,y)\cdot K(s,x,y)dydx = (IV).$$

Likewise, we have $(VI) = (VIII)$ and $(III) = (VII)$. Thus, we have

$$\Big[(I) + (II) + \cdots + (IX)\Big] = 2\Big[(IV) + (VI) + (III)\Big]$$

$$= 2\Big[\int_{S^s}\int_{D^s_+} (*)dydx + \int_{S^s}\int_{D^s_-} (*)dydx + \int_{D^s_+}\int_{D^s_-} (*)dydx\Big]$$

$$= 2\Big[\int_{S^s}\int_{D^s_+ \cup D^s_-} (*)dydx + \int_{D^s_+}\int_{D^s_-} (*)dydx\Big]$$

$$= 2\Big[\int_{S^s}\int_{\mathbb{R}^N} (*)dydx + \int_{D^s_+}\int_{D^s_-} (*)dydx\Big] = 2[(D) + (B)]$$

where the third equality follows $\int_{S^s}\int_{S^s}(*)dydx = (V) = 0$.

In order to use the lower bound of the kernel (3), we need to restrict the above integral on a subset of $\{|x-y| \leq \zeta\}$. For this purpose, we define the subsets $\tilde{D}^s_+, \tilde{D}^s_-$ and $\tilde{S}^s$ by $\tilde{D}^s_\pm = D^s_\pm \cap B(ar)$ and $\tilde{S}^s = S_s \cap B(ar)$. Then, if $x, y \in B(ar)$, then $|x - y| \leq 2ar \leq 2a \leq \zeta$ from (29). Thus, from the lower bound of the kernel (3), we have

$$(D) = \int_{S^s}\int_{\mathbb{R}^N} (*)dydx$$

$$= \int_{S^s}\int_{\mathbb{R}^N} \Big[(0 - 0) - \Big(1_{D^s_+}(y) - 1_{D^s_-}(y)\Big)\Big]\Big(-\varphi(s, y)\Big) \cdot K(s, x, y)dydx$$

$$= \int_{S^s}\int_{\mathbb{R}^N} |\varphi(s, y)| \cdot K(s, x, y)dydx$$

$$\geq \Lambda^{-1}\int_{\tilde{S}^s}\int_{B(ar)} \frac{|\varphi(s, y)|}{|x - y|^{N+\alpha}}dydx \geq \Lambda^{-1}(2ar)^{-(N+\alpha)}\int_{\tilde{S}^s} \|\varphi(s, \cdot)\|_{L^1(B(ar))}dx$$

$$= \Lambda^{-1} \cdot (2ar)^{-(N+\alpha)} \cdot |\tilde{S}^s| \cdot \|\varphi(s, \cdot)\|_{L^1(B(ar))} \geq \Lambda^{-1} \cdot (2ar)^{-(N+\alpha)} \cdot |\tilde{S}^s| \cdot \frac{8}{10}$$

where, for the last equality, the estimate (30) was used.

Also, we have

$$(B) = \int_{D^s_+}\int_{D^s_-} (*)dydx$$

$$= \int_{D^s_+}\int_{D^s_-} \Big[\Big(1_{D^s_+}(x) - 0\Big) - \Big(0 - 1_{D^s_-}(y)\Big)\Big]\Big(\varphi(s, x) - \varphi(s, y)\Big) \cdot K(s, x, y)dydx$$

$$= 2\int_{D^s_+}\int_{D^s_-} \Big(\varphi(s, x) - \varphi(s, y)\Big) \cdot K(s, x, y)dydx$$

$$= 2\Big[\int_{D^s_+}\int_{D^s_-} \varphi(s, x) \cdot K(s, x, y)dydx + \int_{D^s_+}\int_{D^s_-} -\varphi(s, y) \cdot K(s, x, y)dydx\Big]$$

$$= 2[(B_1) + (B_2)].$$

We can obtain the following lower bound of (B_1) by the following way:

$$(B_1) = \int_{D_+^s} \int_{D_-^s} \varphi(s,x) \cdot K(s,x,y) dy dx = \int_{D_-^s} \int_{D_+^s} \varphi(s,x) \cdot K(s,x,y) dx dy$$

$$\geq \Lambda^{-1} \int_{\tilde{D}_-^s} \int_{\tilde{D}_+^s} \frac{\varphi(s,x)}{|x-y|^{N+\alpha}} \cdot dx dy \geq \Lambda^{-1} (2ar)^{-(N+\alpha)} \int_{\tilde{D}_-^s} \|\varphi^+(s,\cdot)\|_{L^1(B(ar))} dy$$

$$\geq \Lambda^{-1} (2ar)^{-(N+\alpha)} \cdot |\tilde{D}_-^s| \cdot \frac{3}{10}.$$

Likewise, for (B_2), we have $(B_2) \geq \Lambda^{-1} \cdot (2ar)^{-(N+\alpha)} \cdot |\tilde{D}_+^s| \cdot \frac{3}{10}$.

Now we have a desirable estimate for $\left[(I) + (II) + \cdots + (IX) \right]$:

$$\frac{1}{2} \left[(I) + (II) + \cdots + (IX) \right] = \left[(IV) + (VI) + (III) \right] = [(D) + (B)]$$

$$\geq \left[\Lambda^{-1} \cdot (2ar)^{-(N+\alpha)} \cdot |\tilde{S}^s| \cdot \frac{8}{10} \right.$$

$$\left. + 2 \left(\Lambda^{-1} \cdot (2ar)^{-(N+\alpha)} \cdot |\tilde{D}_-^s| \cdot \frac{3}{10} + \Lambda^{-1} \cdot (2ar)^{-(N+\alpha)} \cdot |\tilde{D}_+^s| \cdot \frac{3}{10} \right) \right]$$

$$\geq \Lambda^{-1} \cdot (2ar)^{-(N+\alpha)} \cdot \frac{6}{10} \cdot \left(|\tilde{S}^s| + |\tilde{D}_-^s| + |\tilde{D}_+^s| \right)$$

$$\geq \Lambda^{-1} \cdot (2ar)^{-(N+\alpha)} \cdot \frac{6}{10} \cdot \left((ar)^N \cdot V_N \right) = C_2 \cdot r^{-\alpha}.$$

It prove the claim (28), under the assumption $\|\varphi(t)\|_{L^1(\mathbb{R}^N)} \geq \frac{9}{10}$, because

$$\|\varphi(t)\|_{L^1(\mathbb{R}^N)} = \|\varphi(0)\|_{L^1(\mathbb{R}^N)} + \int_0^t \frac{d}{ds} \|\varphi(s,\cdot)\|_{L^1(\mathbb{R}^N)} ds$$

$$\leq 1 + C_2 \cdot r^{-\alpha} \cdot t \quad \text{for any } t \in [0, \delta_2 r^\alpha].$$

On the other hand, if $\|\varphi(t)\|_{L^1(\mathbb{R}^N)} \leq \frac{9}{10}$, then we have

$$\|\varphi(t)\|_{L^1(\mathbb{R}^N)} \leq \frac{9}{10} \leq (1 - C_2 \cdot \frac{t}{r^\alpha})$$

as long as $t \leq \frac{1}{10} C_2^{-1} r^\alpha$. Therefore, by taking $\delta_{L^1} := \min\{\delta_2, \frac{1}{10} C_2^{-1}\}$ and $C_{L^1} := C_2$, we finish the proof of Lemma 3.5.

$\square$

STEP 5. Combining all conditions.

Now we are ready to finish the proof of the main proposition 3.1. In STEP 2-4, we proved that

$$(31) \qquad \int_{\mathbb{R}^N} |\varphi(s,x)| |x|^\gamma dx \leq r^\gamma (1 + C_{conc} A^{\frac{\alpha-\gamma}{N}} \frac{s}{r^\alpha}) \quad \text{for } s \in [0, T),$$

$$(32) \quad \|\varphi(s,\cdot)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{A}{r^N} (1 - C_{L^\infty} A^{\frac{\alpha}{N}} \frac{s}{r^\alpha}) \quad \text{for } s \in [0, \min\{\delta_{L^\infty} \cdot r^\alpha, T\}], \text{ and}$$

$$(33) \qquad \|\varphi(s,\cdot)\|_{L^1(\mathbb{R}^N)} \leq (1 - C_{L^1} \cdot \frac{s}{r^\alpha}) \quad \text{for } s \in [0, \min\{\delta_{L^1} \cdot r^\alpha, T\}].$$

Note that the constants C_{L^1}, C_{L^∞}, and C_{conc} are independent of A as long as $A \geq 1$ while δ_{L^1} and δ_{L^∞} depend on A. We define $\delta_3 := \min\{\delta_{L^1}, \delta_{L^\infty}\}$ so that the above three estimates (31), (32), and (33) hold at the same time for all $s \in [0, \min\{\delta_3 r^\alpha, T\}]$. Without loss of generality, we can assume $C_{L^1} = C_{L^\infty} \leq C_{conc}$.

Recall that we are looking for $\beta > 0$ and $z(r, s)$ such that $\varphi(s) \in \left(\frac{r}{z}\right)^\beta \mathcal{U}_z$. Thus, from Definition 2.3 of $\mathcal{U}_r$ and from the above three estimates (31), (32), and (33), we need the followings:

$$
(34) \quad
\begin{aligned}
r^\gamma(1 + C_{conc}A^{\frac{\alpha-\gamma}{N}}\frac{s}{r^\alpha}) &\leq \left(\frac{r}{z}\right)^\beta z^\gamma, \\
\frac{A}{r^N}(1 - C_{L^\infty}A^{\frac{\alpha}{N}}\frac{s}{r^\alpha}) &\leq \left(\frac{r}{z}\right)^\beta \cdot \frac{A}{z^N}, \text{ and} \\
(1 - C_{L^1}\cdot\frac{s}{r^\alpha}) &\leq \left(\frac{r}{z}\right)^\beta.
\end{aligned}
$$

Remark 3.4. (34) is equivalent to

$$(35) \qquad r(1 + C_{conc}A^{\frac{\alpha-\gamma}{N}}\frac{s}{r^\alpha})^{1/(\gamma-\beta)} \leq z,$$

$$(36) \qquad z \leq r(1 - C_{L^\infty}A^{\frac{\alpha}{N}}\frac{s}{r^\alpha})^{-1/(N+\beta)}, \text{ and}$$

$$(37) \qquad z \leq r(1 - C_{L^1}\cdot\frac{s}{r^\alpha})^{-1/\beta}.$$

The power of A in (35) is strictly less than that of A in (36). This fact is crucial because we can choose A large enough to hold (35) and (36) at the same time. Then we can make β small enough to hold (37), too. We will now give all the details.

We take any $A \geq 1$ large enough to satisfy the inequality:

$$\frac{8}{\gamma}(N + (1/2)) \cdot C_{conc}A^{\frac{\alpha-\gamma}{N}} \leq C_{L^\infty}A^{\frac{\alpha}{N}}.$$

In addition, we take any $\beta \in (0, \gamma/2]$ so small that the following inequality holds:

$$\frac{4}{\gamma}\beta \cdot C_{conc}A^{\frac{\alpha-\gamma}{N}} \leq C_{L^1}.$$

Finally, we define a constant L by

$$L := \frac{2}{\gamma - \beta} \cdot C_{conc}A^{\frac{\alpha-\gamma}{N}}$$

and a function $z(r, s)$ by

$$z(r, s) := r(1 + L\frac{s}{r^\alpha}).$$

For the Concentration-condition, from (II) of Lemma 3.2, we have

$$\int_{\mathbb{R}^N} |\varphi(s, x)||x|^\gamma dx \leq r^\gamma(1 + C_{conc}A^{\frac{\alpha-\gamma}{N}}\frac{s}{r^\alpha}) = r^\gamma\left[1 + \frac{\gamma-\beta}{2} \cdot L \cdot \frac{s}{r^\alpha}\right]$$

$$\leq r^\gamma\left[1 + L \cdot \frac{s}{r^\alpha}\right]^{\gamma-\beta} = \left(\frac{r}{z}\right)^\beta \cdot z^\gamma$$

where the last inequality holds as long as for $s \leq (1/L)r^\alpha$. We define $\delta_4 := \min\{\delta_3, (1/L)\}$.

On the other hand, from $0 < \beta \le \gamma/2 < 1/2$, we observe the followings:

$$(38) \qquad 2 \cdot (N + \beta) \cdot L \le \frac{8}{\gamma}(N + (1/2)) \cdot C_{conc} A^{\frac{\alpha-\gamma}{N}} \le C_{L^\infty} A^{\frac{\alpha}{N}} \quad \text{and}$$

$$(39) \qquad\qquad\qquad \beta \cdot L \le \frac{4}{\gamma}\beta \cdot C_{conc} A^{\frac{\alpha-\gamma}{N}} \le C_{L^1}.$$

For the L^∞-condition, from (38) and from (I) and (IV) of Lemma 3.2, we have

$$\|\varphi(s,\cdot)\|_{L^\infty(\mathbb{R}^N)} \le \frac{A}{r^N}(1 - C_{L^\infty} A^{\frac{\alpha}{N}}\frac{s}{r^\alpha}) \le \frac{A}{r^N} \cdot \left[1 - 2 \cdot (N+\beta) \cdot L \cdot \frac{s}{r^\alpha}\right]$$

$$\le \frac{A}{r^N} \cdot \left[1 + 2 \cdot (N+\beta) \cdot L \cdot \frac{s}{r^\alpha}\right]^{-1} \le \frac{A}{r^N} \cdot \left[1 + L \cdot \frac{s}{r^\alpha}\right]^{-(N+\beta)} = \left(\frac{r}{z}\right)^\beta \cdot \frac{A}{z^N}$$

for $s \in [0, \min\{\delta_5 r^\alpha, T\}]$ where $\delta_5 := \min\{\delta_4, (1/L) \cdot C\}$.

For the L^1-condition, from (39) and (I) and (III) of Lemma 3.2, we have

$$\|\varphi(s,\cdot)\|_{L^1(\mathbb{R}^N)} \le (1 - C_{L^1}\frac{s}{r^\alpha}) \le 1 - \beta \cdot L \cdot \frac{s}{r^\alpha} = \left(\frac{r}{z}\right)^\beta$$

for $s \in [0, \min\{\delta_5 r^\alpha, T\}]$.

Together with the mean zero property of φ in STEP 1, we proved for any $s \in [0, \min\{\delta_5 r^\alpha, T\}]$ with $r \in (0, 1]$ and for any $\varphi_0 \in \mathcal{U}_r$, we have the evolution estimate

$$\varphi(s) \in \left(\frac{r}{r(1 + L\frac{s}{r^\alpha})}\right)^\beta \mathcal{U}_{r(1 + L\frac{s}{r^\alpha})} = \left(\frac{r}{z}\right)^\beta \mathcal{U}_z.$$

which proves (21).

It remains to prove (22). Let $r = 1$ (i.e. $\varphi_0 \in \mathcal{U}_1$). Note that Lemma 3.3 holds for all time $s \in [0, T)$ and L^p norm is decreasing all time $s \in [0, T)$ and for any $1 \le p \le \infty$ from (II) of Lemma 2.3. Thus we have $\varphi(s) \in (1 + Ls)\mathcal{U}_1$ for all $s \in [0, T)$. This is the end of the proof of Proposition 3.1.

$\square$

4. PROOF OF THE PART (II) OF THEOREM 1.1

Proof of the part (II) of Theorem 1.1. Let t be any time between 0 and T. Thanks to (II) of Lemma 2.4 and (II) of Lemma 2.3, the only thing we need to do is to find an estimate on $r^{-\beta}\left|\int_{\mathbb{R}^3} w(t,x)\varphi_0(x)dx\right|$ for $\varphi_0 \in \mathcal{S} \cap \mathcal{U}_r$ with $0 < r \le 1$. From Proposition 3.1, we have a smooth solution φ on $[0, t]$ correspoding to the initial data φ_0 with the kernel $K^{(t)}$, which is defined by $K^{(t)}(s) := K(t - s)$ (see the definition (20)). From Lemma 2.5, we want a control on $r^{-\beta}\left|\int_{\mathbb{R}^3} w_0(x)\varphi(t,x)dx\right|$. Indeed,

(40)

$$\|w(t,\cdot)\|_{C^\beta(\mathbb{R}^N)} \le C \cdot \left(\|w(t,\cdot)\|_{L^\infty} + \sup_{\varphi_0 \in \mathcal{S} \cap \mathcal{U}_r, 0 < r \le 1} r^{-\beta}\left|\int_{\mathbb{R}^3} w(t,x)\varphi_0(x)dx\right|\right)$$

$$\le C \cdot \left(\|w_0\|_{L^\infty} + \sup_{\varphi_0 \in \mathcal{S} \cap \mathcal{U}_r, 0 < r \le 1} r^{-\beta}\left|\int_{\mathbb{R}^3} w_0(x)\varphi(t,x)dx\right|\right).$$

Remark 4.1. The main idea is to repeat (21) as many time as we want until the time evolution reaches the given time $t \in (0, T)$. For example, as long as $s_1 \leq \delta r^\alpha$, $s_2 \leq \delta z(r, s_1)^\alpha$, and $z(r, s_1) \leq 1$, we can repeat Proposition 3.1 twice to get the following time evolution:

$$\varphi(0) \in \mathcal{U}_r \Rightarrow \varphi(s_1) \in \left(\frac{r}{z(r, s_1)}\right)^\beta \mathcal{U}_{z(r, s_1)}$$

$$\Rightarrow \varphi(s_1 + s_2) \in \left(\frac{r}{z(r, s_1)}\right)^\beta \times \left(\frac{z(r, s_1)}{z(z(r, s_1), s_2)}\right)^\beta \mathcal{U}_{z(z(r, s_1), s_2)}$$

$$= \left(\frac{r}{z(z(r, s_1), s_2)}\right)^\beta \mathcal{U}_{z(z(r, s_1), s_2)}.$$

However, when $z(r, s)$ reaches 1 before the given time t, then we cannot use (21) any more. Instead, we need to use (22), which grows as time increases. For this reason, we obtain only (42) first which depends on t. This defect is overcome by investigating the evolution of the L^1 norm of $\varphi(s)$ (see (44)). Since this examination requires a careful estimate (43) for repetitions of (21), we present a rigours argument below. As a result, the final estimate is independent of the length of time interval (see (46)).

Define a constant $\eta := (1 + L \cdot \delta) > 1$. For each $r \in (0, 1]$, we define the integer $k = k(r) \geq 1$ such that $r \cdot \eta^{k-1} \leq 1 < r \cdot \eta^k$. Also define $z_n = z_n(r)$ for $n = 0, 1, 2, \cdots, k-1, k$ by

$$z_n = \begin{cases} r \cdot \eta^n & \text{if } 0 \leq n \leq (k-1), \\ 1 & \text{if } n = k. \end{cases}$$

Note that $r = z_0 < z_1 < \cdots < z_{k-1} \leq 1 = z_k$.

We find $\tilde{t} = \tilde{t}(r) \in [0, \delta r^\alpha (\eta^\alpha)^{k-1}) = [0, \delta(z_{k-1})^\alpha)$ such that $z_{k-1}(1 + L\frac{\tilde{t}}{(z_{k-1})^\alpha})$
$= 1$, which is always possible because $z_{k-1} \leq 1 < z_{k-1} \cdot (1 + L\frac{\delta(z_{k-1})^\alpha}{(z_{k-1})^\alpha}) = z_{k-1} \cdot \eta$.

Also define $t_n = t_n(r)$ for $n = 0, 1, 2, \cdots, k-1, k, k+1$ by

$$t_n = \begin{cases} \delta \cdot r^\alpha \left(\frac{(\eta^\alpha)^n - 1}{\eta^\alpha - 1}\right) & \text{if } 0 \leq n \leq (k-1), \\ (t_{k-1} + \tilde{t}) & \text{if } n = k, \\ \infty & \text{if } n = k+1. \end{cases}$$

Note that, for $1 \leq n \leq (k-1)$,

$$t_n = \delta r^\alpha \left(1 + \eta^\alpha + (\eta^\alpha)^2 + \cdots + (\eta^\alpha)^{n-1}\right)$$

$$= \delta r^\alpha + \delta r^\alpha \eta^\alpha + \delta r^\alpha (\eta^\alpha)^2 + \cdots + \delta r^\alpha (\eta^\alpha)^{n-1}$$

$$= \delta(z_0)^\alpha + \delta(z_1)^\alpha + \cdots + \delta(z_{n-1})^\alpha \text{ and}$$

$$t_n - t_{n-1} = \delta(z_{n-1})^\alpha.$$

Now we make a partition of $[0, \infty) \subset \mathbb{R}^1$ by

$$[0, \infty) = [t_0, t_1) \cup [t_1, t_2) \cup \cdots \cup [t_{k-2}, t_{k-1}) \cup [t_{k-1}, t_k) \cup [t_k, t_{k+1})$$

where these union are disjoint.

Finally, we are ready to apply the main proposition 3.1 as many time as we want. Indeed, if $t \in [t_n, t_{n+1})$ with $0 \le n \le (k-1)$, then we can repeat the main proposition 3.1 so that we obtain

$$\varphi(t) \in \left(\frac{r}{z_1}\right)^\beta \times \left(\frac{z_1}{z_2}\right)^\beta \times \cdots \times \left(\frac{z_n}{z_n(1 + L \cdot \frac{t-t_n}{(z_n)^\alpha})}\right)^\beta \mathcal{U}_{z_n(1+L\cdot\frac{t-t_n}{(z_n)^\alpha})}$$

$$= \left(\frac{r}{z_n(1 + L \cdot \frac{t-t_n}{(z_n)^\alpha})}\right)^\beta \mathcal{U}_{z_n(1+L\cdot\frac{t-t_n}{(z_n)^\alpha})}.$$

Moreover, because

(41) $$\varphi(t_k) \in \left(\frac{r}{z_k}\right)^\beta \mathcal{U}_{z_k} = \left(\frac{r}{1}\right)^\beta \mathcal{U}_1,$$

we get, for the case $t \in [t_k, t_{k+1}) = [t_k, \infty)$,

$$\varphi(t) \in \left(\frac{r}{1}\right)^\beta \cdot (1 + L(t - t_k)) \cdot \mathcal{U}_1.$$

From the above argument, for any fixed $r \in (0, 1]$, we can extend the function $z = z(r, s)$ of Proposition 3.1 up to all $s \in [0, \infty)$ by

$$z(r, s) = \begin{cases} z_n(1 + L \cdot \frac{s-t_n}{(z_n)^\alpha}) & \text{for } s \in [t_n, t_{n+1}) \text{ with } 0 \le n \le (k-1), \\ 1 & \text{for } s \in [t_k, t_{k+1}) = [t_k, \infty). \end{cases}$$

In terms of the function z, we obtained

$$\varphi(t) \in \left(\frac{r}{z}\right)^\beta \cdot (1 + L \cdot (t - t_k)^+) \cdot \mathcal{U}_z.$$

As a result, we have

$$\left| \int_{\mathbb{R}^3} w_0(x)\varphi(t, x)dx \right| \le \left(\frac{r}{z}\right)^\beta \cdot (1 + L \cdot t) \cdot z^\beta \cdot \|w_0\|_{C^\beta(\mathcal{R}^N)}$$

$$= r^\beta \cdot (1 + L \cdot t) \cdot \|w_0\|_{C^\beta(\mathcal{R}^N)}.$$

From the observation (40), we have proved, for any $t \in [0, T)$,

(42) $$\|w(t, \cdot)\|_{C^\beta(\mathbb{R}^N)} \le C \cdot (1 + L \cdot t) \cdot \|w_0\|_{C^\beta(\mathbb{R}^N)}$$

where C does not depend on t. Note that this estimate blows up as t goes to infinity.

We can overcome the above blow-up defect by obtaining the evolution of L^1-norm of $\varphi(s)$. Indeed, for the case $t < t_k$, i.e. for $t \in [t_n, t_{n+1})$ with $0 \le n \le (k-1)$, the function $z(r, t)$ is bounded below by $C \cdot t^{1/\alpha}$ where C does not depend on $r \in (0, 1]$. Indeed,

(43)
$$\left(z(r, t)\right)^\alpha = \left(z_n(1 + L \cdot \frac{t - t_n}{(z_n)^\alpha})\right)^\alpha \ge (z_n)^\alpha = (r \cdot \eta^n)^\alpha = r^\alpha \cdot (\eta^\alpha)^n$$

$$\ge \left(\frac{\eta^\alpha - 1}{\eta\delta}\right)\delta r^\alpha \cdot \frac{((\eta^\alpha)^{n+1} - 1)}{(\eta^\alpha - 1)} \ge \left(\frac{\eta^\alpha - 1}{\eta\delta}\right)t_{n+1} \ge \left(\frac{\eta^\alpha - 1}{\eta\delta}\right)t.$$

Recall that $\varphi(t) \in (r/z)^\beta \cdot \mathcal{U}_z$ for any $t < t_k$. Thus, thanks to (43), we have the evolution of L^1-norm $\|\varphi(t)\|_{L^1(\mathbb{R}^N)} \le (r/z)^\beta \le C \cdot r^\beta \cdot t^{-\beta/\alpha}$.

On the other hand, from (41), we have $\|\varphi(t_k)\|_{L^1(\mathbb{R}^N)} \le r^\beta$. Thus, from (II) of

Lemma 2.3, we get $\|\varphi(t)\|_{L^1(\mathbb{R}^N)} \leq r^\beta$ as long as $t \geq t_k$. Therefore, we have a control for any $t \in (0, T)$:

$$
(44) \qquad \left| \int_{\mathbb{R}^3} w_0(x)\varphi(t, x)dx \right| \leq \|w_0\|_{L^\infty(\mathcal{R}^N)} \cdot \|\varphi(t)\|_{L^1(\mathbb{R}^N)}
$$

$$
\leq C \cdot r^\beta \cdot \max\{1, \frac{1}{t^{\beta/\alpha}}\} \cdot \|w_0\|_{L^\infty(\mathcal{R}^N)}.
$$

Thus, from (40), we have

$$
(45) \qquad \|w(t, \cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C \cdot \max\{1, \frac{1}{t^{\beta/\alpha}}\} \cdot \|w_0\|_{L^\infty(\mathbb{R}^N)}
$$

where C does not depend on t. Now we can combine (42) with (45) to get

$$
(46) \qquad \|w(t, \cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C \cdot \min\{\max\{1, \frac{1}{t^{\beta/\alpha}}\}, (1 + L \cdot t)\} \cdot \|w_0\|_{C^\beta(\mathbb{R}^N)}
$$

$$
\leq C \cdot \|w_0\|_{C^\beta(\mathbb{R}^N)} \text{ for any } t \in [0, T).
$$

Similarly, we can prove $\|\varphi(t)\|_{L^\infty(\mathbb{R}^N)} \leq C \cdot r^\beta \cdot \max\{1, \frac{1}{t^{(N+\beta)/\alpha}}\}$. As a result, from (40), we have

$$
(47) \qquad \|w(t, \cdot)\|_{C^\beta(\mathbb{R}^N)} \leq C\Big(\|w_0\|_{L^\infty(\mathbb{R}^N)} + \max\{1, \frac{1}{t^{(N+\beta)/\alpha}}\} \cdot \|w_0\|_{L^1(\mathbb{R}^N)} \Big).
$$

$\square$

5. APPENDIX

5.1. **Proof of the part (I) of Theorem 1.1.**

Proof of the part (I) of Theorem 1.1. In this subsection, we suppose that the kernel K satisfies not the *weak-*$(*)$-kernel condition in Definition 1.2 but the $(*)$-kernel condition in Definition 1.1 (we recall that the latter condition implies the former one). Note that the kernel K does not need to satisfy (10) any more. Thus, we first construct a family of kernels K_ϵ keeping all the parameters of the $(*)$-kernel condition uniformly in $\epsilon > 0$, and satisfying (10). Then we use the conclusion of the part (II) of Theorem 1.1.

We define Φ by $\Phi(t, x, y) := \Phi^1(t)\Phi^2(x)\Phi^2(y)$ for $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^N$ and $\Phi_\epsilon(\cdot) := \epsilon^{-(2N+1)}\Phi(\cdot/\epsilon)$ where Φ^1 and Φ^2 are standard C_c^∞ mollifiers in $\mathbb{R}$ and $\mathbb{R}^N$, respectively. Let $(w_0)_\epsilon := w_0 * \Phi^2$ and $h(t, x, y) := k(t, x, y - x)$. Then we define a family of kernels by $h_\epsilon := h^\epsilon *_{t,x,y} \Phi_\epsilon$ where

$$
h^\epsilon(t, x, y) := \begin{cases} h(t, x, y) & \text{for } |x - y| < (1/\epsilon) \text{ with } t \in [0, \min\{(1/\epsilon), T\}], \\ \Lambda^{-1} & \text{otherwise.} \end{cases}
$$

Since $|h^\epsilon(t, x, y)| \leq \Lambda(1 + \epsilon^{-\omega}) < \infty$ for all t, x and y, we observe that $k_\epsilon(t, x, z) := h_\epsilon(t, x, x + z)$ satisfies the condition (10).

For each $\epsilon \ll \zeta/2$, the associated kernel $K_\epsilon(t, x, y) := k_\epsilon(t, x, y - x) \cdot |y - x|^{-(N+\alpha)}$ satisfies the $(*)$-kernel condition on the same parameter set of the original kernel K except $\zeta_\epsilon := \zeta/2$ and $\Lambda_\epsilon := 2\Lambda$ (for $\alpha \geq 1$, we assume further $\epsilon \ll s_0/2$ and $(s_0)_\epsilon := s_0/2$). Then, we can construct a weak solution w_ϵ corresponding the kernel K_ϵ and the initial data $(w_0)_\epsilon$, and this solution w_ϵ is smooth since k_ϵ satisfy (10) (for existence, see [21] or refer the approximation scheme in [4] while smoothness is a consequence of a standard energy argument). Thanks to the part (II) of Theorem

1.1, these solutions satisfy (7), (8), and (9). As a result, we can extract a limit function w, which is a weak solution for the original kernel K and the initial data w_0. □

5.2. **Proof of Theorem 1.2.**

Proof of Theorem 1.2. For convenience, we define a function g by $g(x) = G(x) \cdot |x|^{N+\alpha}$. In addition to all the assumptions of Theorem 1.2, we assume further

$$(48) \qquad \theta_0 \in C^\infty(\overline{\mathbb{R}^N}), \quad g \in C^\infty(\overline{\mathbb{R}^N}), \text{ and } \phi \in C^\infty(\overline{\mathbb{R}}).$$

Then there exists a weak solution θ of (11) in global time and it is smooth. Indeed, for existence issue, we refer to Benilan and Brezis [3] or the appendix in the paper [4]. Smoothness follows a difference quotient argument.

We will show that the conclusions of Theorem 1.2 hold for this smooth solution θ. Moreover, it will be clear that the constants C and β depend only on the parameters in the hypotheses of Theorem 1.2 and they are independent of the actual norms coming from the above additional assumption (48). Thus the conclusions of Theorem 1.2 without (48) follows by a limit argument.

Remark 5.1. Indeed, if we do not have (48), then we regularize θ_0, g, and ϕ first:

$$(\theta_0)_\epsilon := \theta_0 * \Phi_\epsilon^2, \quad g_\epsilon := g^\epsilon * \Phi_\epsilon^2, \quad \text{and } \phi_\epsilon := \phi * \Phi_\epsilon^1$$

where Φ^1 and Φ^2 are mollifiers in $\mathbb{R}^1$ and $\mathbb{R}^N$, respectively, and g^ϵ is defined by
$$g^\epsilon(x) := \begin{cases} g(x) \text{ if } |x| \leq (1/\epsilon), \\ 0 \text{ otherwies.} \end{cases}$$ As a result, we obtain (48) for $(\theta_0)_\epsilon, g_\epsilon$, and ϕ_ϵ.
Moreover, for any $\epsilon \leq (\zeta/2)$, all the assumptions (the parameters) of Theorem 1.2 still work for for $(\theta_0)_\epsilon, g_\epsilon$, and ϕ_ϵ except we need to replace the original ζ by $\zeta/2$ for the condition (12).

We take a derivative $(D_e\theta := w)$ on the equation (11) so that we get the following equation

$$\partial_t w(t,x) - \int_{\mathbb{R}^N} (w(t,y) - w(t,x))\phi''(\theta(t,y) - \theta(t,x))G(y-x)dy = 0.$$

By putting $K(t,x,y) := \phi''(\theta(t,y) - \theta(t,x))G(y-x)$, this function $w(= D_e\theta)$ solves the linear equation (1). Moreover, it is easy to see that this new kernel K satisfies (2), (3), and (10) directly (a rigorous proof can be completed by using the difference quotient argument, which is contained in [4]). Then, Theorem 1.2 for the case $\alpha < 1$ follows once we apply the part (II) of Theorem 1.1 to w.

For the the case $\alpha \geq 1$, we need to verify the cancellation condition (5) to get the *weak-(∗)-kernel* condition. Let $s \in (0,1)$, $t \in [0,T]$ and $x \in \mathbb{R}^N$. Then, we have

$$\Big| \int_{S^{N-1}} k(t,x,s\sigma)\sigma d\sigma \Big| = \Big| \int_{S^{N-1}} K(t,x,x+s\sigma)|s\sigma|^{N+\alpha}\sigma d\sigma \Big|$$

$$= \Big| \int_{S_+^{N-1}} \phi''(\theta(t,x+s\sigma) - \theta(t,x))G(s\sigma)s^{N+\alpha}\sigma d\sigma$$

$$+ \int_{S_-^{N-1}} \phi''(\theta(t,x+s\sigma) - \theta(t,x))G(s\sigma)s^{N+\alpha}\sigma d\sigma \Big|$$

where S_+^{N-1} and S_-^{N-1} are upper and lower hemispheres, respectively. Then, by symmetry of $G(\cdot)$,

$$= \Big| \int_{S_+^{N-1}} \Big[\phi''(\theta(t,x+s\sigma) - \theta(t,x)) - \phi''(\theta(t,x-s\sigma) - \theta(t,x)) \Big] G(s\sigma)s^{N+\alpha}\sigma d\sigma.$$

We use the assumption $\phi'' \in C^\nu$:

$$\leq \int_{S_+^{N-1}} [\phi'']_{C^\nu(\mathbb{R})} \cdot \Big| \theta(t,x+s\sigma) - \theta(t,x-s\sigma) \Big|^\nu G(s\sigma)s^{N+\alpha}|\sigma|d\sigma$$

$$\leq \int_{S_+^{N-1}} [\phi'']_{C^\nu(\mathbb{R})} \cdot \|\nabla\theta(t)\|_{L_x^\infty}^\nu \Big| 2s\sigma \Big|^\nu G(s\sigma)s^{N+\alpha}d\sigma$$

$$\leq C[\phi'']_{C^\nu(\mathbb{R})} \cdot \|\nabla\theta_0\|_{L^\infty}^\nu \cdot \sqrt{\Lambda} \cdot s^\nu \cdot \int_{S_+^{N-1}} (1+s^\omega)d\sigma$$

$$\leq C \cdot M \cdot \sqrt{\Lambda} \cdot s^\nu \cdot (1+s^\omega) \leq C \cdot M \cdot \sqrt{\Lambda} \cdot s^\nu$$

where the proof of non-increasing of $\|\nabla\theta(t)\|_{L_x^\infty}$ is in the part (II) of Lemma 2.3. By putting $\tau := C \cdot M \cdot \sqrt{\Lambda}$ with $s_0 := 1$, we get the condition (5). Then, we apply the part (II) of Theorem 1.1 to w.

$\square$

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